DISSERTATION

Supergravity in Two Spacetime Dimensions

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By

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Abstract

Supersymmetry is an essential component of all modern theories such as strings, branes and supergravity, because it is considered as an indispensable ingredient in the search for a unified field theory. Lower dimensional models are of particular importance for the investigation of physical phenomena in a somewhat simplified context. Therefore, two-dimensional supergravity is discussed in the present work.

The constraints of the superfield method are adapted to allow for supergravity with bosonic torsion. As the analysis of the Bianchi identities reveals, a new vector superfield is encountered besides the well-known scalar one. The constraints are solved both with superfields using a special decomposition of the supervielbein, and explicitly in terms of component fields in a Wess-Zumino gauge.

The graded Poisson Sigma Model (gPSM) is the alternative method used to construct supersymmetric gravity theories. In this context the graded Jacobi identity is solved algebraically for general cases. Some of the Poisson algebras obtained are singular, or several potentials contained in them are restricted. This is discussed for a selection of representative algebras. It is found, that the gPSM is far more flexible and it shows the inherent ambiguity of the supersymmetric extension more clearly than the superfield method. Among the various models spherically reduced Einstein gravity and gravity with torsion are treated. Also the Legendre transformation to eliminate auxiliary fields, superdilaton theories and the explicit solution of the gPSM equations of motion for a typical model are presented.

Furthermore, the PSM field equations are analyzed in detail, leading to the so called "symplectic extension". Thereby, the Poisson tensor is extended to become regular by adding new coordinates to the target space. For gravity models this is achieved with one additional coordinate.

Finally, the relation of the gPSM to the superfield method is established by extending the base manifold to become a supermanifold.

Contents

Chapter 1

Introduction

The study of diffeomorphism invariant theories in $1+1$ dimensions has for quite some time been a fertile ground for acquiring some insight into the unsolved problems of quantum gravity in higher dimensions. Indeed, the whole field of spherically symmetric gravity belongs to this class, from d-dimensional Einstein theory to extended theories like the Jordan-Brans-Dicke theory [1–5] or 'quintessence' [6–11] which may now seem to obtain observational support [12–17]. Also, equivalent formulations for 4d Einstein theory with nonvanishing torsion ('teleparallelism' [18–29]) and alternative theories including curvature and torsion [30] are receiving increasing attention.

On the other hand, supersymmetric extensions of gravity [31–35] are believed to be a necessary ingredient for a consistent solution of the problem of quantizing gravity, especially within the framework of string/brane theory [36–38]. These extensions so far are based upon bosonic theories with vanishing torsion.

Despite the fact that so far no tangible direct evidence for supersymmetry has been discovered in nature, supersymmetry [39–42] managed to retain continual interest within the aim to arrive at a fundamental 'theory of everything' ever since its discovery: first in supergravity [31–35] in $d = 4$, then in generalizations to higher dimensions of higher N [43], and finally incorporated as a low energy limit of superstrings [44] or of even more fundamental theories [45] in 11 dimensions.

Even before the advent of strings and superstrings the importance of studies in $1+1$ 'spacetime' had been emphasized [46] in connection with the study of possible superspace formulations [47]. To the best of our knowledge, however, to this day no attempt has been made to generalize the supergravity formulation of (trivial) Einstein-gravity in $d = 2$ to the consideration of two-dimensional $(1, 1)$ supermanifolds for which the condition of vanishing (bosonic) torsion is removed. Only attempts to formulate theories with higher powers of curvature (at vanishing torsion) seem to exist [48]. There

seem to be only very few exact solutions of supergravity in $d = 4$ as well [49–53].

Especially at times when the number of arguments in favour of the existence of an, as yet undiscovered, fundamental theory increase [43] it may seem appropriate to also exploit—if possible—*all* (super-)geometrical generalizations of the two-dimensional stringy world sheet. Actually, such an undertaking can be (and indeed is) successful, as suggested by the recent much improved insight, attained for all (non-supersymmetric) two-dimensional diffeomorphism invariant theories, including dilaton theory, and also comprising torsion besides curvature [54, 55] in the most general manner [56–63]. In the absence of matter-fields (non-geometrical degrees of freedom) all these models are integrable at the classical level and admit the analysis of all global solutions [64, 65, 59–61]. Integrability of two-dimensional gravity coupled to chiral fermions was demonstrated in [66–68]. Even the general aspects of quantization of any such theory now seem to be well understood [58, 69–71, 56, 57]. By contrast, in the presence of matter and if singularities like black holes occur in such models, integrable solutions are known only for very few cases. These include interactions with fermions of one chirality [66] and, if scalar fields are present, only the dilaton black hole [72–75] and models which have asymptotical Rindler behaviour [76]. Therefore, a supersymmetric extension of the matterless case suggests that the solvability may carry over, in general. Then 'matter' could be represented by superpartners of the geometric bosonic field variables.

In view of this situation it seems surprising that the following problem has not been solved so far:

Given a general geometric action of pure gravity in two spacetime dimensions of the form (cf. [77–79] and references therein)

$$L^{\mathrm{gr}} = \int d^2x \sqrt{-g} F(R, \tau^2), \tag{1.1}$$

what are its possible supersymmetric generalizations?

In two dimensions there are only two algebraic-geometric invariants of curvature and torsion, denoted by R and τ^2, respectively, and F is some sufficiently well-behaved function of these invariants. For the case that F does not depend on its second argument, R is understood to be the Ricci scalar of the torsion-free Levi-Civita connection.

As a prototype of a theory with dynamical torsion we may consider the specification of (1.1) to the Katanaev-Volovich (KV) model [54, 55], quadratic in curvature and torsion. Even for this relatively simple particular case of (1.1), a supergravity generalization has not been presented to this day.

The bosonic theory (1.1) may be reformulated as a first order gravity action (FOG) by introducing auxiliary fields ϕ and X^a (the standard momenta

in a Hamiltonian reformulation of the model; cf. [56, 57] for particular cases and [79] for the general discussion)

$$L^{\mathrm{FOG}} = \int_{\mathcal{M}} \phi d\omega + X_a De^a + \epsilon v(\phi, Y) \tag{1.2}$$

where $Y = X^2/2 \equiv X^a X_a/2$ and v is some two-argument function of the indicated variables. In (1.2) e^a is the zweibein and $\omega_{ab} = \omega\epsilon_{ab}$ the Lorentz or spin connection, both 1-form valued, and $\epsilon = \frac{1}{2}e^a \wedge e^b \epsilon_{ba} = ed^2x$ is the two-dimensional volume form ($e = \det(e_m{}^a)$). The torsion 2-form is $De^a = de^a + e^b \wedge \omega\epsilon_b{}^a$.

The function $v(\phi, Y)$ is the Legendre transform [79] of $F(R, \tau^2)$ with respect to the *three* arguments R and τ^a, or, if F depends on the Levi-Civita curvature R only, with respect to this single variable (v depending only on ϕ then). In view of its close relation to the corresponding quantity in generalized dilaton theories which we recall below, we shall call ϕ the 'dilaton' also within the action (1.2).[1] The equivalence between (1.1) and the action (1.2) holds at a global level, if there is a globally well-defined Legendre transform of F. Prototypes are provided by quadratic actions, i.e. R^2-gravity and the model of [54, 55]. Otherwise, in the generic case one still has local (patchwise) equivalence. Only theories for which v or F do not have a Legendre transform even locally, are not at all covered by the respective other formulation. In any case, as far as the supersymmetrization of 2d gravity theories is concerned, we will henceforth focus on the family of actions given by (1.2).

The FOG formulation (1.2) also covers general dilaton theories in two dimensions [59–61, 80–84] ($\widetilde{R}$ is the torsion free curvature scalar),

$$L^{\mathrm{dil}} = \int d^2x \sqrt{-g} \left[\frac{\widetilde{R}}{2}\phi - \frac{1}{2} Z(\phi)(\partial^n \phi)(\partial_n \phi) + V(\phi) \right]. \tag{1.3}$$

Indeed, by eliminating X^a and the torsion-dependent part of ω in (1.2) by their algebraic equations of motion, for regular 2d spacetimes ($e = \sqrt{-g} \neq 0$) the theories (1.3) and (1.2) are locally and globally equivalent if in (1.2) the 'potential' is chosen as [62, 63] (cf. also Section 4.5.3 below for some details as well as [78] for a related approach)

$$v^{\mathrm{dil}}(\phi, Y) = YZ(\phi) + V(\phi). \tag{1.4}$$

There is also an alternative method for describing dilaton gravity by means of an action of the form (1.2), namely by using the variables e^a as a zweibein for a metric $\bar{g}$, related to g in (1.3) according to $g_{mn} = \Omega(\phi)\bar{g}_{mn}$ for a suitable choice of the function Ω (it is chosen in such a way that after

[1] In the literature also $\Phi = -\frac{1}{2}\ln\phi$ carries this name. This definition is useful when ϕ is restricted to $\mathbb{R}_+$ only, as it is often the case for specific models.

transition from the Einstein-Cartan variables in (1.2) and upon elimination of X^a one is left with an action for $\bar{g}$ of the form (1.3) with $\bar{Z} = 0$, i.e. without kinetic term for the dilaton, cf. e.g. [80, 82, 85]).[2] This formulation has the advantage that the resulting potential v depends on ϕ only. It has to be noted, however, that due to a possibly singular behavior of Ω (or $1/\Omega$) the global structures of the resulting spacetimes (maximally extended with respect to $\bar{g}$ versus g) in part are quite different. Moreover, the change of variables in a path integral corresponding to the 'torsion' description of dilaton theories (Y-dependent potential (1.4)) seems advantageous over the one in the 'conformal' description. In this description even interactions with (scalar) matter can be included in a systematic perturbation theory, starting from the (trivially) exact path integral for the geometric part (1.2) [71, 86, 87]. Therefore when describing dilaton theories within the present thesis we will primarily focus on potentials (1.3), linear in Y.

In any case there is thus a huge number of 2d gravity theories included in the present framework. We select only a few for illustrative purposes. One of these is spherically reduced Einstein gravity (SRG) from d dimensions [88–91]

$$Z_{\text{SRG}} = -\frac{d-3}{(d-2)\phi}, \qquad V_{\text{SRG}} = -\lambda^2 \phi^{\frac{d-4}{d-2}}\,, \tag{1.5}$$

where λ is some constant. In the 'conformal approach' mentioned above the respective potentials become

$$Z_{\overline{\text{SRG}}} = 0, \qquad V_{\overline{\text{SRG}}} = -\frac{\lambda^2}{\phi^{\frac{1}{d-2}}}. \tag{1.6}$$

The KV-model, already referred to above, results upon

$$Z_{\text{KV}} = \alpha, \qquad V_{\text{KV}} = \frac{\beta}{2}\phi^2 - \Lambda, \tag{1.7}$$

where Λ, α and β are constant. Two other particular examples are the so-called Jackiw-Teitelboim (JT) model [92–96] with vanishing torsion in (1.2) and no kinetic term of ϕ in (1.3),

$$Z_{\text{JT}} = 0, \qquad V_{\text{JT}} = -\Lambda\phi, \tag{1.8}$$

and the string inspired dilaton black hole (DBH) [72–74] (cf. also [75, 97–104])

$$Z_{\text{DBH}} = -\frac{1}{\phi}, \qquad V_{\text{DBH}} = -\lambda^2\phi, \tag{1.9}$$

[2]Some details on the two approaches to general dilaton gravity may also be found in [79].

which, incidentally, may also be interpreted as the formal limit $d \to \infty$ of (1.5).

For later purposes it will be crucial that (1.2) may be formulated as a Poisson Sigma Model (PSM) [57, 59, 105, 106] (cf. also [107–111]). Collecting zero form and one-form fields within (1.2) as

$$(X^i) := (\phi, X^a), \qquad (A_i) = (dx^m A_{mi}(x)) := (\omega, e_a), \tag{1.10}$$

and after a partial integration, the action (1.2) may be rewritten identically as

$$L^{\text{PSM}} = \int_{\mathcal{M}} dX^i \wedge A_i + \frac{1}{2}\mathcal{P}^{ij} A_j \wedge A_i, \tag{1.11}$$

where the matrix $\mathcal{P}^{ij}$ may be read off by direct comparison. The basic observation in this framework is that this matrix defines a Poisson bracket on the space spanned by coordinates X^i, which is then identified with the target space of a Sigma Model. In the present context this bracket $\{X^i, X^j\} := \mathcal{P}^{ij}$ has the form

$$\{X^a, \phi\} = X^b \epsilon_b{}^a, \tag{1.12}$$

$$\{X^a, X^b\} = v(\phi, Y)\epsilon^{ab}, \tag{1.13}$$

where (throughout this thesis) $Y \equiv \frac{1}{2}X^a X_a$. This bracket may be verified to obey the Jacobi identity.

The gravitational origin of the underlying model is reflected by the first set of brackets: It shows that ϕ is the generator of Lorentz transformations (with respect to the bracket) on the target space $\mathbb{R}^3$. The form of the second set of brackets is already completely determined by the following: Antisymmetry of the bracket leads to proportionality to the ϵ-tensor, while the Jacobi identity for the bracket requires v to be a function of the Lorentz invariant quantities ϕ and X^2 only.

Inspection of the local symmetries of a general PSM,

$$\delta X^i = \mathcal{P}^{ij}\epsilon_j, \qquad \delta A_i = -d\epsilon_i - (\partial_i \mathcal{P}^{jk})\epsilon_k A_j, \tag{1.14}$$

shows that the Lorentz symmetry of the bracket gives rise to the *local* Lorentz symmetry of the gravity action (1.2) (specialization of (1.14) to an ϵ with only nonzero ϕ component, using the identification (1.10)). On the other hand, the second necessary ingredient for the construction of a gravity action, diffeomorphism invariance, is automatically respected by an action of the form (1.11). (It may be seen that the diffeomorphism invariance is also encoded *on-shell* by the remaining two local symmetries (1.14), cf. e.g. [59, 112]).

PSMs relevant for 2d gravity theories (without further gauge field interactions) possess one 'Casimir function' $c(X)$ which is characterized by

the vanishing of the Poisson brackets $\{X^i, c\}$. Different constant values of c characterize symplectic leaves [113, 114]. In the language of gravity theories, for models with asymptotic Minkowski behavior, c is proportional to the ADM mass of the system.[3]

To summarize, the gravity models (1.11), and thus implicitly also any action of the form (1.2) and hence generically of (1.1), may be obtained from the construction of a Lorentz invariant bracket on the two-dimensional Minkowski space $\mathbb{R}^2$ spanned by X^a, with ϕ entering as an additional parameter.[4] The resulting bracket as well as the corresponding models are seen to be parametrized by one two-argument function v in the above way.

The Einstein-Cartan formulation of 2d gravities as in (1.2) or, even more so, in the PSM form (1.11) will turn out to be particularly convenient for obtaining the most general supergravities in $d = 2$. While the metrical formulation of gravity due to Einstein in $d = 4$ appeared very cumbersome for a supersymmetric generalization, the Einstein-Cartan approach appeared to be best suited for the needs of introducing additional fermionic degrees of freedom to pure gravity [31–35].

We now briefly digress to the corresponding strategy of constructing a supersymmetric extension of a gravity theory in a spacetime of general dimension d. By adding to the vielbein ${e_m}^a$ and the Lorentz connection ${\omega_{ma}}^b$ appropriate terms containing a fermionic spin-vector ${\psi_m}^\alpha$, the Rarita-Schwinger field, an action invariant under local supersymmetry can be constructed, where ${\psi_m}^\alpha$ plays the role of the gauge field for that symmetry. In this formulation the generic local infinitesimal supersymmetry transformations are of the form

$$\delta {e_m}^a = -2i(\epsilon\gamma^a\psi_m), \qquad \delta {\psi_m}^\alpha = -D_m\epsilon^\alpha + \cdots \tag{1.15}$$

with $\epsilon = \epsilon(x)$ arbitrary.

In the course of time various methods were developed to make the construction of supergravity actions more systematical. One of these approaches, relying on superfields [117–122], extends the Einstein-Cartan formalism by adding anticommuting coordinates to the spacetime manifold, thus making it a supermanifold, and, simultaneously, by enlarging the structure group with a spinorial representation of the Lorentz group. This method adds many auxiliary fields to the theory, which can be eliminated by choosing

[3] We remark that an analogous conservation law may be established also in the presence of additional matter fields [115, 116].

[4] Actually, this point of view was already used in [57] so as to arrive at (1.11), but without fully realizing the relation to (1.1) at that time. Let us remark on this occasion that in principle one might also consider theories (1.11) with X^a replaced e.g. by $X^a \cdot f(\phi, X^2)$. For a nonvanishing function f this again yields a PSM after a suitable reparametrization of the target space. Also, the identification of the gauge fields A_i in (1.10) could be modified in a similar manner. Hence we do not have to cover this possibility explicitly in what follows. Nevertheless, it could be advantageous to derive by this means a more complicated gravity model from a simpler PSM structure.

appropriate constraints on supertorsion and supercurvature and by choosing a Wess-Zumino type gauge. It will be the approach used in the first part of this thesis (Chapter 2).

The other systematic approach to construct supergravity models for general d is based on the similarity of gravity to a gauge theory. The vielbein and the Lorentz-connection are treated as gauge fields on a similar footing as gauge fields of possibly additional gauge groups. Curvature and torsion appear as particular components of the total field strength.[5] By adding fermionic symmetries to the gravity gauge group, usually taken as the Poincaré, de Sitter or conformal group, one obtains the corresponding supergravity theories [124].

In the two-dimensional case, the supergravity multiplet was first constructed using the superfield approach [46]. Based on that formalism, it was straightforward to *formulate* a supersymmetric generalization of the dilaton theory (1.3), cf. [125]. Before that the supersymmetric generalization of the particular case of the Jackiw-Teitelboim or de Sitter model [92–96] had been achieved within this framework in [126]. Up to global issues, this solved implicitly also the problem of a supersymmetrization of the theories (1.1) in the *torsion-free* case.

Still, the supergravity multiplet obtained from the set of constraints used in [46] consists of the vielbein, the Rarita-Schwinger field and an auxiliary scalar field, but the Lorentz-connection is lost as independent field. It is expressed in terms of the vielbein and the Rarita-Schwinger field. Without a formalism using an independent Lorentz-connection the construction of supersymmetric versions of general theories of the $F(R, \tau^2)$-type is impossible. A straightforward approach consists of a repetition of the calculation of [46] while relaxing the original superspace constraints to allow for a nonvanishing bosonic torsion.

As in higher dimensional theories, the gauge theoretic approach provides a much simpler method for supersymmetrization than the superfield approach. However, it is restricted to relatively simple Lagrangians such as the one of the Jackiw-Teitelboim model (1.8) [127, 128]. The generic model (1.2) or also (1.3) cannot be treated in this manner.

On the other hand, first attempts showed that super dilaton theories may fit into the framework of 'nonlinear' supergauge theories [129, 130], and the action for a super dilaton theory was obtained (without superfields) by a nonlinear deformation of the graded de Sitter group using free differential algebras in [131].

Recently, it turned out [132] (but cf. also [129, 130]) that the framework of PSMs [57, 105, 106], now with a graded target space, represents a very

[5] Note that nevertheless standard gravity theories cannot be just reformulated as YM gauge theories with all symmetries being incorporated in a principle fiber bundle description; one still has to deal with the infinite-dimensional diffeomorphism group (cf. also [123] for an illustration).

direct formalism to deal with super dilaton theories. In particular, it allowed for a simple derivation of the general solution of the corresponding field equations, and in this process yielded the somewhat surprising result that, *in the absence of additional matter fields*, the supersymmetrization of the dilaton theories (1.3) is on-shell trivial. By this we mean that, up to the choice of a gauge, in the general solution to the field equations all fermionic fields can be made to vanish identically by an appropriate choice of gauge while the bosonic fields satisfy the field equations of the purely bosonic theory and are still subject to the symmetries of the latter. This local on-shell triviality of the supersymmetric extension may be interpreted superficially to be yet another consequence of the fact that, from the Hamiltonian point of view, the 'dynamics' of (1.11) is described by just one variable (the Casimir function) which does not change when fermionic fields are added. This type of triviality will cease to prevail in the case of additional matter fields (as is already obvious from a simple counting of the fields and local symmetries involved). Furthermore, the supersymmetrization may be used [125] as a technical device to prove positive energy theorems for supersymmetric *and* non-supersymmetric dilaton theories. Thus, the (local) on-shell triviality of pure 2d supergravity theories by no means implicitly demolishes all the possible interest in their supersymmetric generalizations. This applies similarly to the $F(R, \tau^2)$-theories and to the FOG formulation (1.2) in which we are primarily interested.

Graded PSMs (gPSMs) turn out also to provide a unifying and most efficient framework for the *construction* of supersymmetric extensions of a two-dimensional gravity theory, at least as far as theories of the initially mentioned type (1.1) are considered. This route, sketched already briefly in [132], will be followed in detail within the present thesis.

The main idea of this approach will be outlined in Section 4.1.1. It will be seen that within this framework the problem for a supersymmetric extension of a gravity theory (1.1) is reduced to a finite dimensional problem: Given a Lorentz invariant Poisson bracket on a two-dimensional Minkowski space (which in addition depends also on the 'dilaton' or, equivalently, on the generator of Lorentz transformations ϕ), one has to extend this bracket consistently and in a Lorentz covariant manner to the corresponding superspace.

In spirit this is closely related to the analogous extension of Lie algebras to superalgebras [133–136]. In fact, in the particular case of a linear dependence of v in (1.13) the original Poisson bracket corresponds to a three-dimensional Lie algebra, and likewise any *linear* extension of this Poisson algebra corresponds to a superalgebra. Here we are dealing with general nonlinear Poisson algebras, particular cases of which can be interpreted as finite W-algebras (cf. [137]). Due to that nonlinearity the analysis necessary for the fermionic extension is much more involved and there is a higher ambiguity in the extension (except if one considers this only modulo arbitrary

(super)diffeomorphisms). Therefore we mainly focus on an $N = 1$ extension within this thesis.

Section 2.1 is devoted to the general definitions of superspace used in our present work. A decomposition of the supervielbein useful for solving the supergravity constraints is presented in Section 2.2. In Section 2.3 the supergravity model of Howe [46] is given. The new supergravity constraints for a supergravity model in terms of superfields with independent Lorentz connection are derived in Section 2.4 and the constraints are solved in Section 2.5.

In Chapter 3 the PSM approach is presented for the bosonic case. Beside the known parts (Section 3.1 and 3.2) we present a general method to solve the PSM field equations (Section 3.3). We also show a new 'symplectic extension' (Section 3.4–3.6).

After recapitulating some material on gPSMs in Section 4.1.2 and also setting our notation and conventions, the solution of the ϕ-components of the Jacobi identities is given in Section 4.2.1 simply by writing down the most general Lorentz covariant ansatz for the Poisson tensor. In Section 4.2.2 the remaining Jacobi identities are solved in full generality for nondegenerate and degenerate $N = 1$ fermionic extensions.

The observation that a large degree of arbitrariness is present in these extensions is underlined also by the study of target space diffeomorphisms in Section 4.3. We also point out the advantages of this method in the quest for new algebras and corresponding gravity theories.

In Section 4.4 we shall consider particular examples of the general result. This turns out to be superior to performing a general abstract discussion of the results of Section 4.2. The more so, because fermionic extensions of specific bosonic 2d gravity theories, which have been discussed already in the literature, can be investigated. Supersymmetric extensions of the KV-model (1.7) as compared to SRG (1.5) will serve for illustrative purposes.

The corresponding actions and their relation to the initial problem (1.1) are given in Section 4.6. Also the general relation to the supersymmetric dilatonic theories (1.3) will be made explicit using the results of Section 4.5. Several different supersymmetrizations (one of which is even parity violating) for the example of SRG are compared to the one provided previously in the literature [125]. For each model the corresponding supersymmetry is given explicitly.

In Section 4.7 the explicit solution for a supergravity theory with the bosonic part corresponding to v^{dil} in (1.4) is given.

In Chapter 5 we discuss the relations between the superfield approach of Chapter 2 and the gPSM supergravity theories of Chapter 4.

In the final Chapter 6 we will summarize our findings and comment on possible further investigations.

Appendix A and B define notations and summarize useful identities.

Chapter 2

Supergravity with Superfields

Supergravity can be formulated in superspace, where the two-dimensional x-space of pure gravity is enlarged by fermionic (anticommutative) θ-variables. On this underlying superspace a gravity theory comprising Einstein-Cartan variables $E_M{}^A(x,\theta)$ and $\Omega_{MA}{}^B(x,\theta)$ is established. The basic properties of supergeometry are given in Section 2.1. As target space group the direct sum of a vector and a spinor representation of the Lorentz group is chosen (cf. (2.35) and (2.36) below). In order to reduce the large number of component fields, the coefficients in the θ-expansion of the superfields, constraints in superspace are imposed. The choice of the supergravity constraints is a novel one. We start with the original constraints of the $N = 1$ supergravity model of Howe [46], shortly reviewed in Section 2.3, but in order to retain an independent Lorentz connection $\omega_m(x)$ these constraints have to be modified. A short derivation leading to the new constraints is given in Section 2.4. Then, in Section 2.5, the new supergravity model based on the new set of constraints is investigated. The solution in terms of superfields as well as the component field expansion in a Wess-Zumino type gauge are given. It turns out that the more general new supergravity model contains in addition to the well-known supergravity multiplet $\{e_m{}^a, \psi_m{}^\alpha, A\}$, where $e_m{}^a$ is the zweibein, $\psi_m{}^\alpha$ the Rarita-Schwinger field and A the auxiliary field, a new connection multiplet $\{k^a, \varphi_m{}^\alpha, \omega_m\}$ consisting of a vector field k^a, a further spin-vector $\varphi_m{}^\alpha$ and the Lorentz connection ω_m.

2.1 Supergeometry

Although the formulae of supergeometry often look quite similar to the ones of pure bosonic geometry, some attention has to be devoted to a convenient definition e.g. of the signs. The difference to ordinary gravity becomes obvious when Einstein-Cartan variables are extended to superspace. As the

structure group a direct product of a vector and a spinor representation of the Lorentz group is needed. This leads to a rich structure for supertorsion and supercurvature components and for the corresponding Bianchi identities.

2.1.1 Superfields

In $d = 2$ we consider a superspace with two commuting (bosonic) and two anticommuting (Grassmann or spinor) coordinates $z^M = \{x^m, \theta^\mu\}$ where lower case Latin ($m = 0, 1$) and Greek indices ($\mu = 1, 2$) denote commuting and anticommuting coordinates, respectively:

$$z^M z^N = z^N z^M (-1)^{MN}. \tag{2.1}$$

Within our conventions for Majorana spinors (cf. Appendix B) the first anticommuting element of the Grassmann algebra is supposed to be real, $(\theta^+)^* = \theta^+$, while the second one is purely imaginary, $(\theta^-)^* = -\theta^-$ (cf. Appendix B).

Our construction is based on differential geometry of superspace. We shall not deal with subtle mathematical definitions [120, 121]. We just set our basic conventions.

In superspace right and left derivatives have to be distinguished. The relation between the partial derivatives

$$\vec{\partial}_M \equiv \frac{\vec{\partial}}{\partial z^M}, \qquad \overleftarrow{\partial}_M \equiv \frac{\overleftarrow{\partial}}{\partial z^M}, \tag{2.2}$$

which act to right and to the left, respectively, becomes

$$\vec{\partial}_M f = f \overleftarrow{\partial}_M (-1)^{M(f+1)}, \tag{2.3}$$

where in the exponent M and f are 1 for anticommuting quantities and 0 otherwise. For our purpose it is sufficient to follow one simple working rule allowing to generalize ordinary formulae of differential geometry to superspace. Any vectorfield in superspace

$$\vec{V} = V^M \vec{\partial}_M \tag{2.4}$$

is invariant under arbitrary nondegenerate coordinate changes $z^M \to \bar{z}^M(z)$:

$$V^M \vec{\partial}_M = \bar{V}^M \frac{\vec{\partial} z^L}{\partial \bar{z}^M} \frac{\vec{\partial} \bar{z}^N}{\partial z^L} \vec{\partial}_{\bar{N}} \tag{2.5}$$

Summation over repeated indices is assumed, and derivatives are always supposed to act *to the right* from now. So we drop the arrows in the sequel. From (2.5) follows our simple basic rule: Any formula of differential geometry in ordinary space can be taken over to superspace if the summation is

always performed from the upper left corner to the lower right one with no indices in between ('ten to four'), and the order of the indices in each term of the expression must be the same. Otherwise an appropriate factor (-1) must be included.

The components of differential superforms of degree p are defined by

$$\Phi = \frac{1}{p!} dz^{M_p} \wedge \cdots \wedge dz^{M_1} \Phi_{M_1 \cdots M_p} \tag{2.6}$$

and the exterior derivative by

$$d\Phi = \frac{1}{p!} dz^{M_p} \wedge \cdots \wedge dz^{M_1} \wedge dz^N \partial_N \Phi_{M_1 \cdots M_p}. \tag{2.7}$$

A simple calculation shows that as a consequence of the Leibniz rule for the partial derivative and of (2.7) the rule of the exterior differential acting on a product of a q-superform Ψ and a p-superform Φ becomes

$$d(\Psi \wedge \Phi) = \Psi \wedge d\Phi + (-1)^p d\Psi \wedge \Phi. \tag{2.8}$$

Thus we arrive at the simple prescription that d effectively acts *from the right*. This should not be confused with the *partial derivative* in our convention acting to the right.

Each superfield $S(x, \theta)$ can be expanded in the anticommutative variable (θ-expansion)

$$S(x, \theta) = s(x) + \theta^\lambda s_\lambda(x) + \frac{1}{2}\theta^2 s_2(x), \tag{2.9}$$

so that the coefficients s, s_λ and s_2 are functions of the commutative variable x^m only. For the zeroth order in θ the shorthand $S| = s$ is utilized. If s is invertible, then

$$S^{-1} = \frac{1}{s} - \frac{1}{s^2}\theta^\lambda s_\lambda - \frac{1}{2}\theta^2 \left(\frac{1}{s^3} s^\lambda s_\lambda + \frac{1}{s^2} s_2 \right) \tag{2.10}$$

is the inverse of the superfield S.

For a matrix-valued superfield

$$A = a + \theta^\lambda a_\lambda + \frac{1}{2}\theta^2 a_2, \tag{2.11}$$

where the coefficients a, a_λ and a_2 are matrices of equal dimensions, a^{-1} exists and A has definite parity $p(A) = p(a) = p(a_\lambda) + 1 = p(a_2)$, the inverse reads

$$A^{-1} = a^{-1} - a^{-1}(\theta^\lambda a_\lambda)a^{-1} - \frac{1}{2}\theta^2 \,(a^{-1}a^\lambda a^{-1} a_\lambda a^{-1} + a^{-1} a_2 a^{-1}). \tag{2.12}$$

The Taylor expansion in terms of the θ-variables of a function $V(S)$, where S is of the form (2.9), is given by

$$V(S) = V(s) + \theta^\lambda s_\lambda V'(s) + \frac{1}{2}\theta^2 \left[s_2 V'(s) - \frac{1}{2} s^\lambda s_\lambda V''(s) \right]. \tag{2.13}$$

2.1.2 Superspace Metric

The invariant interval reads

$$ds^2 = dz^M \otimes dz^N G_{NM} = dz^M \otimes dz^N G_{MN} (-1)^{MN}, \tag{2.14}$$

where G_{MN} is the superspace metric. This metric can be used to lower indices of a vector field,

$$V_M = V^N G_{NM} = G_{MN} V^N (-1)^N. \tag{2.15}$$

The generalization to an arbitrary tensor is obvious. Defining the inverse metric according to the rule

$$V^M = G^{MN} V_N = V_N G^{NM} (-1)^N, \tag{2.16}$$

and demanding that sequential lowering and raising indices shall be the identical operation yields the main property of the inverse metric

$$G^{MN} G_{NP} = \delta_P{}^M (-1)^{MP} = \delta_P{}^M (-1)^M = \delta_P{}^M (-1)^P. \tag{2.17}$$

The last identities follow from the diagonality of the Kronecker symbol $\delta_P{}^M = \delta_P^M$. Thus the inverse metric is not an inverse matrix in the usual sense. From (2.15) the quantity

$$V^2 = V^M V_M = V_M V^M (-1)^M \tag{2.18}$$

is a scalar, (but e.g. $V_M V^M$ is not!).

2.1.3 Linear Superconnection

We assume that our superspace is equipped with a Riemann-Cartan geometry that is with a metric and with a metrical connection $\Gamma_{MN}{}^P$. The latter defines the covariant derivative of a tensor field. Covariant derivatives of a vector V^N and covector V_N read as

$$\nabla_M V^N = \partial_M V^N + V^P \Gamma_{MP}{}^N (-1)^{PM}, \tag{2.19}$$
$$\nabla_M V_N = \partial_M V_N - \Gamma_{MN}{}^P V_P. \tag{2.20}$$

The metricity condition for the metric is

$$\nabla_M G_{NP} = \partial_M G_{NP} - \Gamma_{MN}{}^R G_{RP} - \Gamma_{MP}{}^R G_{RN} (-1)^{NP} = 0. \tag{2.21}$$

The action of an (anti)commutator of covariant derivatives,

$$[\nabla_M, \nabla_N\} = \nabla_M \nabla_N - \nabla_N \nabla_M (-1)^{MN} \tag{2.22}$$

on a vector field (2.15),

$$[\nabla_M, \nabla_N\} V_P = -R_{MNP}{}^R V_R - T_{MN}{}^R \nabla_R V_P, \tag{2.23}$$

is defined in terms of curvature and torsion:

$$R_{MNP}{}^R = \partial_M \Gamma_{NP}{}^R - \Gamma_{MP}{}^S \Gamma_{NS}{}^R (-1)^{N(S+P)} - (M \leftrightarrow N)(-1)^{MN}, \tag{2.24}$$

$$T_{MN}{}^R = \Gamma_{MN}{}^R - \Gamma_{NM}{}^R (-1)^{MN} \tag{2.25}$$

2.1.4 Einstein-Cartan Variables

In our construction we use Cartan variables: the superspace vierbein $E_M{}^A$ and the superconnection $\Omega_{MA}{}^B$. Capital Latin indices from the beginning of the alphabet ($A = a, \alpha$) transform under the Lorentz group as a vector ($a = 0, 1$) and spinor ($\alpha = 1, 2$), respectively. Cartan variables are defined by

$$G_{MN} = E_M{}^A E_N{}^B \eta_{BA} (-1)^{AN}, \tag{2.26}$$

and the metricity condition is

$$\nabla_M E_N{}^A = \partial_M E_N{}^A - \Gamma_{MN}{}^P E_P{}^A + E_N{}^B \Omega_{MB}{}^A (-1)^{M(B+N)} = 0. \tag{2.27}$$

Raising and lowering of the anholonomic indices $(A, B, \dots)$ is performed by the superspace Minkowski metric

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \epsilon_{\alpha\beta} \end{pmatrix}, \quad \eta^{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \epsilon^{\alpha\beta} \end{pmatrix}, \tag{2.28}$$

consisting of the two-dimensional Minkowskian metric $\eta_{ab} = \eta^{ab} = \mathrm{diag}(+-)$ and $\epsilon_{\alpha\beta}$, the antisymmetric (Levi-Civita) tensor defined in Appendix B. The Minkowski metric and its inverse (2.28) in superspace obey

$$\eta_{AB} = \eta_{BA}(-1)^A, \quad \eta^{AB}\eta_{BC} = \delta_C{}^A (-1)^A. \tag{2.29}$$

The metric (2.28) is invariant under the Lorentz group acting on tensor indices from the beginning of the alphabet. In fact (2.28) is not unique in this respect because $\epsilon_{\alpha\beta}$ may be multiplied by an arbitrary nonzero factor. This may represent a freedom to generalize our present approach. In fact, in order to have a correct dimension of all terms in the line element of superspace, that factor should carry the dimension of length. A specific choice for it presents a freedom in approaches to supersymmetry. In the following this factor will be suppressed. Therefore any apparent differences in dimensions between terms below are not relevant.

The transformation of anholonomic indices $(A, B, \dots)$ into holonomic ones $(M, N, \dots)$ and vice versa is performed using the supervierbein and its inverse $E_A{}^M$ defined as

$$E_A{}^M E_M{}^B = \delta_A{}^B, \quad E_M{}^A E_A{}^N = \delta_M{}^N. \tag{2.30}$$

To calculate the superdeterminant parts of the supervielbein and its inverse are needed:

$$\mathrm{sdet}(E_M{}^A) = \det(E_m{}^a)\det(E_\alpha{}^\mu). \tag{2.31}$$

In the calculations below we have found it extremely convenient to work directly in the anholonomic basis

$$\hat\partial_A := E_A{}^M \partial_M,$$

defined by the inverse supervierbein (for the ordinary zweibein the notation $\partial_a = e_a{}^m \partial_m$ is used).

The anholonomicity coefficients, defined by $[\hat\partial_A, \hat\partial_B] = C_{AB}{}^C \hat\partial_C$, or in terms of differential forms by $dE^A = -C^A$, can be calculated from the supervierbein and its inverse

$$C_{AB}{}^C = (E_A{}^N \partial_N E_B{}^M - (-1)^{AB} E_B{}^N \partial_N E_A{}^M) E_M{}^C. \tag{2.32}$$

The metricity condition (2.27) formally establishes a one-to-one correspondence between the metrical connection $\Gamma_{MN}{}^P$ and the superconnection $\Omega_{MA}{}^B$. Together with (2.21) it implies

$$\nabla_M \eta_{AB} = 0 \tag{2.33}$$

and the symmetry property

$$\Omega_{MAB} + \Omega_{MBA}(-1)^{AB} = 0. \tag{2.34}$$

In general, the superconnection $\Omega_{MA}{}^B$ is not related to Lorentz transformations alone. The Lorentz connection in superspace must have a specific form and is defined (in $d = 2$) by Ω_M, a superfield with one vector index,

$$\Omega_{MA}{}^B = \Omega_M L_A{}^B, \tag{2.35}$$

where

$$L_A{}^B = \begin{pmatrix} \epsilon_a{}^b & 0 \\ 0 & -\frac{1}{2}\gamma^3{}_\alpha{}^\beta \end{pmatrix} \tag{2.36}$$

contains the Lorentz generators in the bosonic and fermionic sectors. Here the factor in front of γ^3 is fixed by the requirement that under Lorentz transformations γ-matrices are invariant under simultaneous rotations of

vector and spinor indices. Definition and properties of γ-matrices are given in Appendix B. $L_A{}^B$ has the properties

$$L_{AB} = -L_{BA}(-1)^A, \quad L_A{}^B L_B{}^C = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & \frac{1}{4}\delta_\alpha{}^\gamma \end{pmatrix}, \quad \nabla_M L_A{}^B = 0. \tag{2.37}$$

The superconnection $\Omega_{MA}{}^B$ in the form (2.35) is very restricted because the original 32 independent superfield components for the Lorentz superconnection reduce to 4. As a consequence, (2.27) with (2.35) also entails restrictions on the metric connection $\Gamma_{MN}{}^P$.

In terms of the connection (2.35) covariant derivatives of a Lorentz supervector read

$$\nabla_M V^A = \partial_M V^A + \Omega_M V^B L_B{}^A, \tag{2.38}$$
$$\nabla_M V_A = \partial_M V_A - \Omega_M L_A{}^B V_B. \tag{2.39}$$

2.1.5 Supercurvature and Supertorsion

The (anti)commutator of covariant derivatives,

$$[\nabla_M, \nabla_N\} V_A = -R_{MNA}{}^B V_B - T_{MN}{}^P \nabla_P V_A, \tag{2.40}$$

is defined by the same expressions for curvature and torsion as given by (2.24) and (2.25), which in Cartan variables become

$$R_{MNA}{}^B = \partial_M \Omega_{NA}{}^B - \Omega_{MA}{}^C \Omega_{NC}{}^B (-1)^{N(A+C)} - (M \leftrightarrow N)(-1)^{MN}, \tag{2.41}$$
$$T_{MN}{}^A = \partial_M E_N{}^A + E_N{}^B \Omega_{MB}{}^A (-1)^{M(B+N)} - (M \leftrightarrow N)(-1)^{MN}. \tag{2.42}$$

In the anholonomic basis (2.42) turns into

$$T_{AB}{}^C = -C_{AB}{}^C + \Omega_A L_B{}^C - (-1)^{AB} \Omega_B L_A{}^C, \tag{2.43}$$

which due to the special form of the Lorentz connection (2.35) yields for the various components

$$T_{\alpha\beta}{}^\gamma = -C_{\alpha\beta}{}^\gamma - \frac{1}{2}\Omega_\alpha \gamma^3{}_\beta{}^\gamma - \frac{1}{2}\Omega_\beta \gamma^3{}_\alpha{}^\gamma, \tag{2.44}$$
$$T_{\alpha\beta}{}^c = -C_{\alpha\beta}{}^c, \tag{2.45}$$
$$T_{a\beta}{}^\gamma = -C_{a\beta}{}^\gamma - \frac{1}{2}\Omega_a \gamma^3{}_\beta{}^\gamma, \tag{2.46}$$
$$T_{\alpha b}{}^c = -C_{\alpha b}{}^c + \Omega_\alpha \epsilon_b{}^c, \tag{2.47}$$
$$T_{ab}{}^\gamma = -C_{ab}{}^\gamma, \tag{2.48}$$
$$T_{ab}{}^c = -C_{ab}{}^c + \Omega_a \epsilon_b{}^c - \Omega_b \epsilon_a{}^c. \tag{2.49}$$

In terms of the Lorentz connection (2.35) the curvature does not contain quadratic terms

$$R_{MNA}{}^{B} = \left(\partial_M \Omega_N - \partial_N \Omega_M (-1)^{MN}\right) L_A{}^B = F_{MN} L_A{}^B, \tag{2.50}$$

where F_{AB} can be calculated in the anholonomic basis by the formulae

$$F_{AB} = \hat{\partial}_A \Omega_B - (-1)^{AB} \hat{\partial}_B \Omega_A - C_{AB}{}^C \Omega_C, \tag{2.51}$$
$$= \nabla_A \Omega_B - (-1)^{AB} \nabla_B \Omega_A + T_{AB}{}^C \Omega_C. \tag{2.52}$$

Ricci tensor and scalar curvature of the manifold are

$$R_{AB} = R_{CAB}{}^C (-1)^{C(A+B+C)} = L_B{}^C F_{CA} (-1)^{AB}, \tag{2.53}$$
$$R = R_A{}^A (-1)^A = L^{AB} F_{BA}. \tag{2.54}$$

2.1.6 Bianchi Identities

The first Bianchi identity $\hat{D} T^A = E^B \wedge R_B{}^A$ reads in component form

$$\Delta_{ABC}{}^D = R_{[ABC]}{}^D, \tag{2.55}$$

where

$$\begin{aligned} \Delta_{ABC}{}^D &:= \nabla_{[A} T_{BC]}{}^D + T_{[AB|}{}^E T_{E|C]}{}^D \\ &= \nabla_A T_{BC}{}^D + \nabla_B T_{CA}{}^D (-1)^{A(B+C)} + \nabla_C T_{AB}{}^D (-1)^{C(A+B)} \\ &+ T_{AB}{}^E T_{EC}{}^D + T_{BC}{}^E T_{EA}{}^D (-1)^{A(B+C)} + T_{CA}{}^E T_{EB}{}^D (-1)^{C(A+B)} \end{aligned} \tag{2.56}$$

and

$$R_{[ABC]}{}^D = F_{AB} L_C{}^D + F_{BC} L_A{}^D (-1)^{A(B+C)} + F_{CA} L_B{}^D (-1)^{C(A+B)}. \tag{2.57}$$

Due to the restricted form of the Lorentz connection (2.35) the bosonic and spinorial parts of $R_{[ABC]}{}^D$ are given by

$$R_{[\alpha\beta\gamma]}{}^d = 0, \tag{2.58}$$
$$R_{[\alpha\beta\gamma]}{}^\delta = -\frac{1}{2} F_{\alpha\beta} \gamma^3{}_\gamma{}^\delta - \frac{1}{2} F_{\beta\gamma} \gamma^3{}_\alpha{}^\delta - \frac{1}{2} F_{\gamma\alpha} \gamma^3{}_\beta{}^\delta, \tag{2.59}$$
$$R_{[\alpha\beta c]}{}^d = F_{\alpha\beta} \epsilon_c{}^d, \tag{2.60}$$
$$R_{[a\beta\gamma]}{}^\delta = -\frac{1}{2} F_{a\beta} \gamma^3{}_\gamma{}^\delta - \frac{1}{2} F_{a\gamma} \gamma^3{}_\beta{}^\delta, \tag{2.61}$$
$$R_{[a\beta c]}{}^d = F_{a\beta} \epsilon_c{}^d - F_{c\beta} \epsilon_a{}^d, \tag{2.62}$$
$$R_{[ab\gamma]}{}^\delta = -\frac{1}{2} F_{ab} \gamma^3{}_\gamma{}^\delta, \tag{2.63}$$
$$R_{[abc]}{}^d = F_{ab} \epsilon_c{}^d + F_{bc} \epsilon_a{}^d + F_{ca} \epsilon_b{}^d, \tag{2.64}$$
$$R_{[abc]}{}^\delta = 0. \tag{2.65}$$

The second Bianchi identity $\hat{D}R_A{}^B = 0$, where the various components are denoted by the symbol Δ_{ABC},

$$\Delta_{ABC} := \nabla_{[A}F_{BC]} + T_{[AB|}{}^D F_{D|C]} \tag{2.66}$$

$$\begin{aligned} &= \nabla_A F_{BC} + \nabla_B F_{CA}(-1)^{A(B+C)} + \nabla_C F_{AB}(-1)^{C(A+B)} \\ &\quad + T_{AB}{}^D F_{DC} + T_{BC}{}^D F_{DA}(-1)^{A(B+C)} + T_{CA}{}^D F_{DB}(-1)^{C(A+B)}, \end{aligned} \tag{2.67}$$

is stated in component form

$$\Delta_{ABC} = 0. \tag{2.68}$$

2.2 Decomposition of the Supervielbein

For the solution of the supergravity constraints it will be very useful to decompose the supervierbein and its inverse in terms of the new superfields $B_m{}^a$, $B_\mu{}^\alpha$, $\Phi_\mu{}^m$ and $\Psi_m{}^\mu$:

$$E_M{}^A = \begin{pmatrix} B_m{}^a & \Psi_m{}^\nu B_\nu{}^\alpha \\ \Phi_\mu{}^n B_n{}^a & B_\mu{}^\alpha + \Phi_\mu{}^n \Psi_n{}^\nu B_\nu{}^\alpha \end{pmatrix} \tag{2.69}$$

$$E_A{}^M = \begin{pmatrix} B_a{}^m + B_a{}^n \Psi_n{}^\nu \Phi_\nu{}^m & -B_a{}^n \Psi_n{}^\mu \\ -B_\alpha{}^\nu \Phi_\nu{}^m & B_\alpha{}^\mu \end{pmatrix} \tag{2.70}$$

The superfields $B_a{}^m$ and $B_\alpha{}^\mu$ are the inverse of $B_m{}^a$ and $B_\mu{}^\alpha$, respectively:

$$B_a{}^m B_m{}^b = \delta_a{}^b, \qquad B_\alpha{}^\mu B_\mu{}^\beta = \delta_\alpha{}^\beta. \tag{2.71}$$

To shorten notation the frequently occurring products of $\Phi_\mu{}^m$ and $\Psi_m{}^\mu$ with the B-fields are abbreviated by

$$\Phi_\nu{}^a = \Phi_\nu{}^m B_m{}^a, \qquad \Phi_\alpha{}^m = B_\alpha{}^\nu \Phi_\nu{}^m, \qquad \Phi_\alpha{}^a = B_\alpha{}^\nu \Phi_\nu{}^m B_m{}^a, \tag{2.72}$$

$$\Psi_a{}^\nu = B_a{}^m \Psi_m{}^\nu, \qquad \Psi_m{}^\alpha = \Psi_m{}^\nu B_\nu{}^\alpha, \qquad \Psi_a{}^\alpha = B_a{}^m \Psi_m{}^\nu B_\nu{}^\alpha. \tag{2.73}$$

The superdeterminant of the supervielbein (2.31) expressed in terms of $B_m{}^a$ and $B_\mu{}^\alpha$ reads

$$\mathrm{sdet}(E_M{}^A) = \frac{\det(B_m{}^a)}{\det(B_\mu{}^\alpha)}. \tag{2.74}$$

Now the anholonomicity coefficients (2.32) in terms of their decomposition (2.69) become

$$C_{\alpha\beta}{}^c = (B_\alpha{}^\mu B_\beta{}^\nu + B_\beta{}^\mu B_\alpha{}^\nu)(\Phi_\mu{}^l \partial_l \Phi_\nu{}^n - \partial_\mu \Phi_\nu{}^n) B_n{}^c, \tag{2.75}$$

$$\begin{aligned} C_{\alpha\beta}{}^\gamma = {} & (B_\alpha{}^\mu B_\beta{}^\nu + B_\beta{}^\mu B_\alpha{}^\nu)(\Phi_\mu{}^l \partial_l \Phi_\nu{}^n - \partial_\mu \Phi_\nu{}^n) \Psi_n{}^\gamma \\ & + (B_\alpha{}^\mu B_\beta{}^\nu + B_\beta{}^\mu B_\alpha{}^\nu)(\Phi_\mu{}^l \partial_l B_\nu{}^\gamma - \partial_\mu B_\nu{}^\gamma), \end{aligned} \tag{2.76}$$

$$
\begin{aligned}
C_{\alpha\beta}{}^{c} = -\Psi_a{}^{\alpha}(B_\alpha{}^{\mu}B_\beta{}^{\nu} + B_\beta{}^{\mu}B_\alpha{}^{\nu})(\Phi_\mu{}^{l}\partial_l\Phi_\nu{}^{n} - \partial_\mu\Phi_\nu{}^{n})B_n{}^{c} \\
- B_a{}^{n}B_\beta{}^{\mu}((\partial_n\Phi_\mu{}^{l})B_l{}^{c} + \Phi_\mu{}^{l}\partial_l B_n{}^{c} - \partial_\mu B_n{}^{c}),
\end{aligned}
\quad (2.77)
$$

$$
\begin{aligned}
C_{\alpha\beta}{}^{\gamma} = -\Psi_a{}^{\alpha}(B_\alpha{}^{\mu}B_\beta{}^{\nu} + B_\beta{}^{\mu}B_\alpha{}^{\nu})(\Phi_\mu{}^{l}\partial_l\Phi_\nu{}^{n} - \partial_\mu\Phi_\nu{}^{n})\Psi_n{}^{\gamma} \\
- \Psi_a{}^{\alpha}(B_\alpha{}^{\mu}B_\beta{}^{\nu} + B_\beta{}^{\mu}B_\alpha{}^{\nu})(\Phi_\mu{}^{l}\partial_l B_\nu{}^{\gamma} - \partial_\mu B_\nu{}^{\gamma}) \\
- B_a{}^{n}B_\beta{}^{\mu}((\partial_n\Phi_\mu{}^{l})\Psi_l{}^{\gamma} + \Phi_\mu{}^{l}\partial_l\Psi_n{}^{\gamma} - \partial_\mu\Psi_n{}^{\gamma} + \partial_n B_\mu{}^{\gamma}),
\end{aligned}
\quad (2.78)
$$

$$
\begin{aligned}
C_{ab}{}^{c} = \Psi_b{}^{\beta}\Psi_a{}^{\alpha}(B_\alpha{}^{\mu}B_\beta{}^{\nu} + B_\beta{}^{\mu}B_\alpha{}^{\nu})(\Phi_\mu{}^{l}\partial_l\Phi_\nu{}^{n} - \partial_\mu\Phi_\nu{}^{n})B_n{}^{c} \\
- (B_a{}^{m}B_b{}^{n} - B_b{}^{m}B_a{}^{n})\Psi_m{}^{\mu}((\partial_n\Phi_\mu{}^{l})B_l{}^{c} + \Phi_\mu{}^{l}\partial_l B_n{}^{c} - \partial_\mu B_n{}^{c}) \\
+ (B_a{}^{m}\partial_m B_b{}^{n} - B_b{}^{m}\partial_m B_a{}^{n})B_n{}^{c},
\end{aligned}
\quad (2.79)
$$

$$
\begin{aligned}
C_{ab}{}^{\gamma} = \Psi_b{}^{\beta}\Psi_a{}^{\alpha}(B_\alpha{}^{\mu}B_\beta{}^{\nu} + B_\beta{}^{\mu}B_\alpha{}^{\nu})(\Phi_\mu{}^{l}\partial_l\Phi_\nu{}^{n} - \partial_\mu\Phi_\nu{}^{n})\Psi_n{}^{\gamma} \\
- (B_a{}^{m}B_b{}^{n} - B_b{}^{m}B_a{}^{n})\Psi_m{}^{\mu}((\partial_n\Phi_\mu{}^{l})\Psi_l{}^{\gamma} + \Phi_\mu{}^{l}\partial_l\Psi_n{}^{\gamma} - \partial_\mu\Psi_n{}^{\gamma} + \partial_n B_\mu{}^{\gamma}) \\
- (B_a{}^{m}B_b{}^{n} - B_b{}^{m}B_a{}^{n})(\partial_m\Psi_n{}^{\gamma}).
\end{aligned}
\quad (2.80)
$$

2.3 Supergravity Model of Howe

Our next task is the recalculation of the two-dimensional supergravity model originally found in [46] and also in [138–141]. It is not our intention to review the steps of calculation in this section, we merely give a summary of the results for reference purposes. Details of the calculation can be found in [141] and in Section 2.5 below, where a more general supergravity model with torsion is considered.

The original supergravity constraints chosen by Howe were

$$
T_{\alpha\beta}{}^{\gamma} = 0, \qquad T_{\alpha\beta}{}^{c} = 2i\gamma^{c}{}_{\alpha\beta}, \qquad T_{ab}{}^{c} = 0. \quad (2.81)
$$

An equivalent set of constraints is

$$
T_{\alpha\beta}{}^{\gamma} = 0, \qquad T_{\alpha\beta}{}^{c} = 2i\gamma^{c}{}_{\alpha\beta}, \qquad \gamma_a{}^{\beta\alpha}F_{\alpha\beta} = 0, \quad (2.82)
$$

as can be seen when inspecting the Bianchi identities.

2.3.1 Component Fields

The constraints (2.81) are identically fulfilled with the expressions for the supervielbein and the Lorentz superconnection below. These expression are obtained using a Wess-Zumino type gauge; details can be found in [141]. We also specify the superfields of the decomposition of the supervielbein

(2.69), which is needed to solve the supergravity constraints with superfield methods (cf. Section 2.5.2 below).

As physical x-space variables one obtains the vielbein $e_m{}^a$, the Rarita-Schwinger field $\psi_m{}^\alpha$ and the auxiliary scalar field A. They constitute the supergravity multiplet $\{e_m{}^a, \psi_m{}^\alpha, A\}$. Although the multiplet was derived using Einstein-Cartan variables in superspace, in the residual x-space no Einstein-Cartan variables were left over. There is no independent Lorentz connection. As a consequence of the constraints (2.81) it is eliminated by the condition $t_{ab}{}^c = 0$, where

$$t_{ab}{}^c = -c_{ab}{}^c + \omega_a \epsilon_b{}^c - \omega_b \epsilon_a{}^c - 2i(\psi_a \gamma^c \psi_b), \tag{2.83}$$

yielding $\omega_a = \breve{\omega}_a$ with

$$\begin{aligned} \breve{\omega}_a &:= \tilde{\omega}_a - i\epsilon^{bc}(\psi_c \gamma_a \psi_b) && (2.84)\\ &= \tilde{\omega}_a - 4i(\psi\gamma^3\lambda_c). && (2.85) \end{aligned}$$

Here $\tilde{\omega}^a = \epsilon^{nm}(\partial_m e_n{}^a)$ is the usual torsion free connection of pure bosonic gravity, and for the second line (B.60) was used. The term quadratic in $\psi_m{}^\alpha$ is necessary to make $\breve{\omega}_a$ covariant with respect to supersymmetry transformations i.e. no derivative of the supersymmetry parameter ϵ^α shows up in its transformation rule.

A similar argument applies to the derivative of the Rarita-Schwinger field

$$\begin{aligned} \breve{\sigma}_\mu &:= \epsilon^{mn}\left(\partial_n \psi_{m\mu} + \frac{1}{2}\breve{\omega}_n(\gamma^3\psi_m)_\mu - \frac{i}{2}A(\gamma_n\psi_m)_\mu\right), && (2.86)\\ &= \epsilon^{nm}\breve{D}_m\psi_{n\mu} + iA(\gamma^3\psi)_\mu, && (2.87) \end{aligned}$$

which is obviously covariant with respect to Lorentz transformations, and also with respect to supersymmetry.

We obtain for the supervielbein

$$E_m{}^a = e_m{}^a + 2i(\theta\gamma^a\psi_m) + \frac{1}{2}\theta^2\left[Ae_m{}^a\right], \tag{2.88}$$

$$E_m{}^\alpha = \psi_m{}^\alpha - \frac{1}{2}\breve{\omega}_m(\theta\gamma^3)^\alpha + \frac{i}{2}A(\theta\gamma_m)^\alpha - \frac{1}{2}\theta^2\left[\frac{3}{2}A\psi_m{}^\alpha + i(\breve{\sigma}\gamma_m\gamma^3)^\alpha\right], \tag{2.89}$$

$$E_\mu{}^a = i(\theta\gamma^a)_\mu, \tag{2.90}$$

$$E_\mu{}^\alpha = \delta_\mu{}^\alpha + \frac{1}{2}\theta^2\left[-\frac{1}{2}A\delta_\mu{}^\alpha\right], \tag{2.91}$$

and for its inverse

$$E_a{}^m = e_a{}^m - i(\theta\gamma^m\psi_a) + \frac{1}{2}\theta^2\left[-2(\psi_a\lambda^m)\right], \tag{2.92}$$

$$\begin{aligned} E_a{}^\mu = &-\psi_a{}^\mu + i(\theta\gamma^b\psi_a)\psi_b{}^\mu + \frac{1}{2}\breve{\omega}_a(\theta\gamma^3)^\mu - \frac{i}{2}A(\theta\gamma_a)^\mu \\ &+ \frac{1}{2}\theta^2\left[2(\psi_a\lambda^b)\psi_b{}^\mu - \frac{i}{2}\breve{\omega}_b(\psi_a\gamma^b\gamma^3)^\mu + i(\breve{\sigma}\gamma_a\gamma^3)^\mu\right], \end{aligned} \tag{2.93}$$

$$E_\alpha{}^m = -i(\theta\gamma^m)_\alpha + \frac{1}{2}\theta^2\left[-2\lambda^m{}_\alpha\right], \tag{2.94}$$

$$E_\alpha{}^\mu = \delta_\alpha{}^\mu + i(\theta\gamma^b)_\alpha\psi_b{}^\mu + \frac{1}{2}\theta^2\left[2\lambda^b{}_\alpha\psi_b{}^\mu - \frac{i}{2}\breve{\omega}_b(\gamma^b\gamma^3)_\alpha{}^\mu - \frac{1}{2}A\delta_\alpha{}^\mu\right]. \tag{2.95}$$

The above expressions for the supervielbein are derived from the decomposition (2.69). The $B_a{}^m$ superfield and its inverse with the zweibein at zeroth order read

$$B_a{}^m = e_a{}^m - 2i(\theta\gamma^m\psi_a) + \frac{1}{2}\theta^2\left[-8(\psi_a\lambda^m) - Ae_a{}^m\right], \tag{2.96}$$

$$B_m{}^a = e_m{}^a + 2i(\theta\gamma^a\psi_m) + \frac{1}{2}\theta^2\left[Ae_m{}^a\right]. \tag{2.97}$$

The $B_\alpha{}^\mu$ and its inverse are given by

$$B_\alpha{}^\mu = \delta_\alpha{}^\mu + i(\theta\gamma^b)_\alpha\psi_b{}^\mu + \frac{1}{2}\theta^2\left[2\lambda^b{}_\alpha\psi_b{}^\mu - \frac{i}{2}\breve{\omega}_b(\gamma^b\gamma^3)_\alpha{}^\mu - \frac{1}{2}A\delta_\alpha{}^\mu\right], \tag{2.98}$$

$$B_\mu{}^\alpha = \delta_\mu{}^\alpha - i(\theta\gamma^b)_\mu\psi_b{}^\alpha + \frac{1}{2}\theta^2\left[-4\lambda^b{}_\mu\psi_b{}^\alpha + \frac{i}{2}\breve{\omega}_b(\gamma^b\gamma^3)_\mu{}^\alpha + \frac{1}{2}A\delta_\mu{}^\alpha\right]. \tag{2.99}$$

We note that the lowest order $\delta_\alpha{}^\mu$ is responsible for the promotion of the coordinates θ^μ to Lorentz spinors. In the superfield $\Phi_\mu{}^m$, the first term inherently carries the superspace structure,

$$\Phi_\mu{}^m = i(\theta\gamma^m)_\mu + \frac{1}{2}\theta^2\left[4\lambda^m{}_\mu\right], \tag{2.100}$$

and $\Psi_m{}^\mu$, whose zeroth component is the Rarita-Schwinger field, reads

$$\begin{aligned} \Psi_m{}^\mu = &\psi_m{}^\mu + i(\theta\gamma^b\psi_m)\psi_b{}^\mu - \frac{1}{2}\breve{\omega}_m(\theta\gamma^3)^\mu + \frac{i}{2}A(\theta\gamma_m)^\mu \\ &+ \frac{1}{2}\theta^2\left[2(\psi_m\lambda^b)\psi_b{}^\mu - \frac{i}{2}\breve{\omega}_b(\psi_m\gamma^b\gamma^3)^\mu - A\psi_m{}^\mu - i(\breve{\sigma}\gamma_m\gamma^3)^\mu\right]. \end{aligned} \tag{2.101}$$

The equally complicated result for the Lorentz superconnection becomes

$$\Omega_a = \breve{\omega}_a - i(\theta\gamma^b\psi_a)\breve{\omega}_b + A(\theta\gamma^3\psi_a) + 2i(\theta\gamma_a\breve{\sigma}) + \frac{1}{2}\theta^2\left[-2(\psi_a\lambda^b)\breve{\omega}_b + 2iA(\psi\gamma^3\lambda_a) + 4(\psi\gamma_a\breve{\sigma}) + \epsilon_a{}^b(\partial_b A)\right], \quad (2.102)$$

$$\Omega_\alpha = -i(\theta\gamma^b)_\alpha\breve{\omega}_b + A(\theta\gamma^3)_\alpha + \frac{1}{2}\theta^2\left[-2\lambda^b{}_\alpha\breve{\omega}_b - 2iA(\gamma^3\psi)_\alpha + 4\breve{\sigma}_\alpha\right], \quad (2.103)$$

and also for world indices

$$\Omega_m = \breve{\omega}_m + 2A(\theta\gamma^3\psi_m) + 2i(\theta\gamma_m\breve{\sigma}) + \frac{1}{2}\theta^2\left[A\breve{\omega}_m - 4(\lambda_m\breve{\sigma}) + \epsilon_m{}^n(\partial_n A)\right], \quad (2.104)$$

$$\Omega_\mu = A(\theta\gamma^3)_\mu. \quad (2.105)$$

Finally, the superdeterminant can be obtained from (2.31) or (2.74):

$$E = e\left(1 - 2i(\theta\psi) + \frac{1}{2}\theta^2\left[A + 2\psi^2 + \lambda^2\right]\right) \quad (2.106)$$

The formulae in this section were cross-checked against a computer calculation for which a symbolic computer algebra program was adapted [142].

2.3.2 Symmetry Transformations

Under a superdiffeomorphism $\delta z^M = -\xi^M(z)$ and a local Lorentz transformation $\delta V^A = V^B L(z) L_B{}^A$ the supervielbein transforms according to

$$\delta E_M{}^A = -\xi^N(\partial_N E_M{}^A) - (\partial_M \xi^N)E_N{}^A + L E_M{}^B L_B{}^A. \quad (2.107)$$

There are $16 + 4 = 20$ x-space transformation parameters in the superfields $\xi^M(x,\theta)$ and $L(x,\theta)$. The $10 + 5 = 15$ gauge fixing conditions (2.189) and (2.190) reduce this number to 5.[1] The remaining symmetries are the x-space diffeomorphism with parameter $\eta^m(x)$, the local Lorentz transformation with parameter $l(x)$ and the local supersymmetry parametrized with $\epsilon^\alpha(x)$. The superfields for x-space diffeomorphism are characterized by

$$\xi^m = \eta^m, \qquad \xi^\mu = 0, \qquad L = 0, \quad (2.108)$$

and local Lorentz transformations by

$$\xi^m = 0, \qquad \xi^\mu = -\frac{1}{2}l(\theta\gamma^3)^\mu, \qquad L = l. \quad (2.109)$$

In addition to the frame rotation $L = l$ there is also a rotation of the coordinate θ^μ as expressed by the term for ξ^μ in (2.109). This is a consequence of

[1] The gauge conditions are indeed the same as the one in Section 2.5.3 below.

the gauge condition $E_\mu{}^\alpha| = \delta_\mu{}^\alpha$ (cf. (2.189)) and leads to the identification of the anticommutative coordinate θ^μ as a Lorentz spinor. The superfield transformation parameters of local supersymmetry are a bit more complicated:

$$\xi^m = -i(\epsilon\gamma^m\theta) + \frac{1}{2}\theta^2\left[2(\epsilon\lambda^m)\right], \tag{2.110}$$

$$\xi^\mu = \epsilon^\mu + i(\epsilon\gamma^b\theta)\psi_b{}^\mu + \frac{1}{2}\theta^2\left[-2(\epsilon\lambda^b)\psi_b{}^\mu + \frac{i}{2}\breve{\omega}_b(\epsilon\gamma^b\gamma^3)^\mu\right], \tag{2.111}$$

$$L = i(\epsilon\gamma^b\theta)\breve{\omega}_b - A(\epsilon\gamma^3\theta) + \frac{1}{2}\theta^2\left[-2(\epsilon\lambda^b)\breve{\omega}_b - 2iA(\epsilon\gamma^3\psi) + 2(\epsilon\breve{\sigma})\right]. \tag{2.112}$$

These expressions are derived by θ-expansion of (2.107), and by taking the gauge conditions (2.189) and (2.190) into account. Details of the calculation can be found in [141].

Similarly, from (2.107) the transformation laws of the physical x-space fields

$$\delta e_a{}^m = 2i(\epsilon\gamma^m\psi_a), \qquad \delta e_m{}^a = -2i(\epsilon\gamma^a\psi_m), \tag{2.113}$$

$$\delta\psi_m{}^\mu = -\left(\partial_m\epsilon^\mu - \frac{1}{2}\breve{\omega}_m(\epsilon\gamma^3)^\mu + \frac{i}{2}A(\epsilon\gamma_m)^\mu\right), \tag{2.114}$$

$$\delta A = -2(\epsilon\gamma^3\breve{\sigma}) \tag{2.115}$$

follow. Also the variation of the dependent Lorentz connection $\breve{\omega}_m$ defined in (2.84) might be of interest:

$$\delta\breve{\omega}_m = -2i(\epsilon\gamma_m\breve{\sigma}) - 2A(\epsilon\gamma^3\psi_m). \tag{2.116}$$

2.3.3 Supertorsion and Supercurvature

The Bianchi identities (2.55) and (2.68) show that as a consequence of the supergravity constraints (2.81) all components of supertorsion and supercurvature depend on only one single superfield S:

$$T_{\alpha\beta}{}^c = 2i\gamma^c{}_{\alpha\beta}, \qquad T_{\alpha\beta}{}^\gamma = 0, \tag{2.117}$$

$$T_{\alpha b}{}^c = 0, \qquad T_{a\beta}{}^\gamma = -\frac{i}{2}S\gamma_{a\beta}{}^\gamma, \tag{2.118}$$

$$T_{ab}{}^c = 0, \qquad T_{ab}{}^\gamma = \frac{1}{2}\epsilon_{ab}\gamma^{3\gamma\delta}\nabla_\delta S, \tag{2.119}$$

and

$$F_{\alpha\beta} = 2S\gamma^3{}_{\alpha\beta}, \tag{2.120}$$

$$F_{a\beta} = -i(\gamma_a\gamma^3)_\beta{}^\gamma\nabla_\gamma S, \tag{2.121}$$

$$F_{ab} = \epsilon_{ab}\left(S^2 - \frac{1}{2}\nabla^\alpha\nabla_\alpha S\right). \tag{2.122}$$

Using the component field expansion of supervielbein and supercurvature stated in Section 2.3.1 one finds

$$S = A + 2(\theta\gamma^3\breve{\sigma}) + \frac{1}{2}\theta^2\left[\epsilon^{mn}(\partial_n\breve{\omega}_m) - A(A + 2\psi^2 + \lambda^2) - 4i(\psi\gamma^3\breve{\sigma})\right]. \tag{2.123}$$

Here $\breve{\omega}_m$ and $\breve{\sigma}_\mu$ are the expressions defined in (2.84) and (2.86).

The superfield S is the only quantity that can be used to build action functionals invariant with respect to local supersymmetry. A general action in superspace invariant with respect to superdiffeomorphisms takes the form

$$L = \int d^2x \underline{d}^2\theta\, E\, F(x,\theta), \tag{2.124}$$

where $E = \mathrm{sdet}(E_M{}^A)$ is the superdeterminant of the supervielbein and $F(x,\theta)$ is an arbitrary scalar superfield. For an explanation of the Berezin integration (2.124) we refer to [117–122]. The superspace actions

$$L_n = \int d^2x \underline{d}^2\theta\, E\, S^n \tag{2.125}$$

constructed with E and S from above (cf. (2.106) and (2.123)) result in

$$L_0 = \int d^2x\, e\left(A + 2\psi^2 + \lambda^2\right), \tag{2.126}$$

$$L_1 = \int d^2x\, \frac{e}{2}\breve{r}, \tag{2.127}$$

$$L_2 = \int d^2x\, e\left(A\breve{r} + 4\breve{\sigma}^2 - A^2(A + 2\psi^2 + \lambda^2)\right), \tag{2.128}$$

where the θ-integration was done, and where $\breve{r} := 2\epsilon^{mn}(\partial_n\breve{\omega}_m)$.

2.4 Lorentz Covariant Supersymmetry

As a preparation for the 'new' supergravity in this section we consider a theory with covariantly constant supersymmetry.

Starting from the supervielbein and superconnection of rigid SUSY we separate the coordinate space and the tangent space by the introduction of the zweibein $e_a{}^m(x)$. This makes the theory covariant with respect to x-space coordinate transformations given by the superspace parameters $\xi^m(x,\theta) = \eta^m(x)$, $\xi^\mu(x,\theta) = 0$ and $L(x,\theta) = 0$. The local Lorentz transformation in the tangent space is given by the superspace parameter $L(x,\theta) = l(x)$. The gauge fixing condition $B_\alpha{}^\mu| = \delta_\alpha{}^\mu$ shows that this local Lorentz transformation is accompanied by a superdiffeomorphism with parameters $\xi^m(x,\theta) = 0$ and $\xi^\mu(x,\theta) = -\frac{1}{2}l(x)(\theta\gamma^3)^\mu$. When one calculates

the transformations under this local Lorentz boost of the various component fields of the supervielbein and superconnection one finds that some components transform into a derivative of the transformation parameter. In order to isolate this inhomogeneous term we introduce a connection field $\omega_m(x)$ with the Lorentz transformation property $\delta\omega_m = -\partial_m l$. In this way the superfields listed below are derived:

$$E_M{}^A = \begin{pmatrix} e_m{}^a & -\frac{1}{2}\omega_m(\theta\gamma^3)^\alpha \\ i(\theta\gamma^a)_\mu & \delta_\mu{}^\alpha \end{pmatrix} \tag{2.129}$$

$$E_A{}^M = \begin{pmatrix} e_a{}^m & \frac{1}{2}\omega_a(\theta\gamma^3)^\mu \\ -i(\theta\gamma^m)_\alpha & \delta_\alpha{}^\mu - \frac{i}{4}\theta^2\omega_n(\gamma^n\gamma^3)_\alpha{}^\mu \end{pmatrix} \tag{2.130}$$

$$B_a{}^m = e_a{}^m \qquad B_m{}^a = e_m{}^a \tag{2.131}$$

$$B_\alpha{}^\mu = \delta_\alpha{}^\mu - \frac{i}{4}\theta^2\omega_n(\gamma^n\gamma^3)_\alpha{}^\mu \qquad B_\mu{}^\alpha = \delta_\mu{}^\alpha + \frac{i}{4}\theta^2\omega_n(\gamma^n\gamma^3)_\mu{}^\alpha \tag{2.132}$$

$$\Phi_\nu{}^n = i(\theta\gamma^n)_\nu \qquad \Psi_n{}^\nu = -\frac{1}{2}\omega_n(\theta\gamma^3)^\nu \tag{2.133}$$

$$\Omega_\alpha = -i\,\omega_n(\theta\gamma^n)_\alpha \qquad \Omega_a = \omega_a \tag{2.134}$$

$$\Omega_\mu = 0 \qquad \Omega_m = \omega_m \tag{2.135}$$

From (2.130) the superspace covariant derivatives $\hat\partial_A = E_A{}^M\partial_M$, now also covariant with respect to local Lorentz transformation,

$$\hat\partial_\alpha = \delta_\alpha{}^\mu\partial_\mu - i(\theta\gamma^m)_\alpha\partial_m - \frac{i}{4}\theta^2\omega_n(\gamma^n\gamma^3)_\alpha{}^\mu\partial_\mu, \tag{2.136}$$

$$\hat\partial_a = e_a{}^m\partial_m + \frac{1}{2}\omega_a(\theta\gamma^3)^\mu\partial_\mu \tag{2.137}$$

are obtained.

The anholonomicity coefficients are derived from (2.32) yielding

$$C_{\alpha\beta}{}^c = -2i\gamma^c{}_{\alpha\beta} - \theta^2\,\tau^c\gamma^3{}_{\alpha\beta}, \tag{2.138}$$

$$C_{\alpha\beta}{}^\gamma = \frac{i}{2}\omega_n(\theta\gamma^n)_\alpha\gamma^3{}_\beta{}^\gamma + \frac{i}{2}\omega_n(\theta\gamma^n)_\beta\gamma^3{}_\alpha{}^\gamma, \tag{2.139}$$

$$C_{a\beta}{}^c = i(\theta\gamma^b)_\beta\,(-c_{ab}{}^c + \omega_a\epsilon_b{}^c), \tag{2.140}$$

$$C_{a\beta}{}^\gamma = -\frac{1}{2}\omega_a\gamma^3{}_\beta{}^\gamma + \frac{i}{8}\theta^2\,r\,\gamma_{a\beta}{}^\gamma, \tag{2.141}$$

$$C_{ab}{}^c = c_{ab}{}^c, \tag{2.142}$$

$$C_{ab}{}^\gamma = \frac{1}{2}f_{ab}(\theta\gamma^3)^\gamma. \tag{2.143}$$

The calculation of the supertorsion is based upon the formulae (2.43),

$$T_{\alpha\beta}{}^{c} = 2i\gamma^{c}{}_{\alpha\beta} + \theta^2\,\tau^c\gamma^3{}_{\alpha\beta}, \tag{2.144}$$
$$T_{\alpha\beta}{}^{\gamma} = 0, \tag{2.145}$$
$$T_{a\beta}{}^{c} = -i(\theta\gamma^b)_\beta\,t_{ab}{}^{c} = +i(\theta\gamma_a\gamma^3)_\beta\tau^c, \tag{2.146}$$
$$T_{a\beta}{}^{\gamma} = -\frac{i}{8}\theta^2\,r\,\gamma_{a\beta}{}^{\gamma}, \tag{2.147}$$
$$T_{ab}{}^{c} = t_{ab}{}^{c}, \tag{2.148}$$
$$T_{ab}{}^{\gamma} = -\frac{1}{2}f_{ab}(\theta\gamma^3)^\gamma, \tag{2.149}$$

and from (2.52) the supercurvature components

$$F_{\alpha\beta} = \frac{1}{2}\theta^2\,r\,\gamma^3{}_{\alpha\beta}, \tag{2.150}$$
$$F_{a\beta} = -if_{ab}(\theta\gamma^b)_\beta = \frac{i}{2}r(\theta\gamma_a\gamma^3)_\beta, \tag{2.151}$$
$$F_{ab} = f_{ab} = \frac{1}{2}\epsilon_{ab}r \tag{2.152}$$

are obtained. For the definition of bosonic torsion and curvature $t_{ab}{}^{c}$ (or τ^c) and f_{ab} (or r) we refer to Appendix A.

The verification of the first and second Bianchi identities can be done by direct calculation and indeed they do not restrict the connection ω_m. The following remarkable formulae are consequences of the Bianchi identities:

$$T_{ab}{}^{c} = -\frac{i}{2}\gamma_b{}^{\beta\alpha}\nabla_\alpha T_{a\beta}{}^{c} \qquad F_{ab} = -\frac{i}{2}\gamma_b{}^{\beta\alpha}\nabla_\alpha F_{a\beta} \tag{2.153}$$
$$T_{a\beta}{}^{c} = \frac{i}{4}(\gamma_a\gamma^3)_\beta{}^{\alpha}\nabla_\alpha(\gamma^{3\delta\gamma}T_{\gamma\delta}{}^{c}) \qquad F_{a\beta} = \frac{i}{4}(\gamma_a\gamma^3)_\beta{}^{\alpha}\nabla_\alpha(\gamma^{3\delta\gamma}F_{\gamma\delta}) \tag{2.154}$$
$$T_{ab}{}^{c} = \frac{1}{8}\epsilon_{ab}\epsilon^{\beta\alpha}\nabla_\alpha\nabla_\beta(\gamma^{3\delta\gamma}T_{\gamma\delta}{}^{c}) \qquad F_{ab} = \frac{1}{8}\epsilon_{ab}\epsilon^{\beta\alpha}\nabla_\alpha\nabla_\beta(\gamma^{3\delta\gamma}F_{\gamma\delta}) \tag{2.155}$$

In this way we get a theory which is covariant with respect to x-space coordinate transformations and local Lorentz transformations. The model also has a further symmetry, namely supersymmetry, given by the superfield parameters $\xi^m(x,\theta) = 0$, $\xi^\mu(x,\theta) = \epsilon^\mu(x)$ and $L(x,\theta) = 0$, but the parameter $\epsilon^\mu(x)$ must be covariantly constant, $\nabla_m\epsilon^\mu = 0$. The main purpose of the present section was the derivation of the supertorsion- and supercurvature components listed above. They are very helpful in finding constraints for the supergeometry of the most general supergravity, where the x-space torsion $t_{ab}{}^{c}$ or curvature f_{ab} are not restricted. We will also see that the original two-dimensional supergravity constraints (2.81) of Howe [46] set the bosonic torsion to zero from the very beginning, and why our first attempt [141] to find a supergravity model with independent connection ω_m could not have been but partially successful.

2.5 New Supergravity

In our generalization we relax the original supergravity constraints (2.81) in order to obtain an independent bosonic Lorentz connection ω_a in the superfield Einstein-Cartan variables $E_A{}^M$ and Ω_A. The first constraint in (2.81) can be left untouched, but we cannot use the third constraint $T_{ab}{}^c = 0$, because this would lead to $t_{ab}{}^c = 0$ as can be seen from (2.148). This observation was the basis of the work in [141], where that third constraint was omitted, leading to an additional Lorentz connection supermultiplet Ω_a, in which the bosonic Lorentz connection was the zeroth component of the θ-expansion, $\omega_a = \Omega_a|$. The drawback of that set of supergravity constraints was the appearance of a second Lorentz connection, dependent on zweibein and Rarita-Schwinger field. Here we choose $\gamma_a{}^{\beta\alpha} F_{\alpha\beta} = 0$ from the alternative set (2.82). This equalizes the Lorentz connection terms in Ω_a and the other superfields $E_a{}^m$ and Ω_α, but does not force the bosonic torsion to zero (cf. also (2.150)). Also the second constraint in (2.81) cannot be maintained. The considerations of the former section showed that this would again lead to a constraint on the bosonic torsion appearing in the θ^2-component of $T_{\alpha\beta}{}^c$ (cf. (2.144)), so we have to weaken that condition and choose the new set of supergravity constraints

$$T_{\alpha\beta}{}^\gamma = 0, \qquad \gamma_a{}^{\beta\alpha} T_{\alpha\beta}{}^c = -4i\delta_a{}^c, \qquad \gamma_a{}^{\beta\alpha} F_{\alpha\beta} = 0. \tag{2.156}$$

The terms proportional to $\gamma^3{}_{\alpha\beta}$ of $T_{\alpha\beta}{}^c$ and $F_{\alpha\beta}$ are a vector superfield K^c and a scalar superfield S, defined according to the ansatz

$$T_{\alpha\beta}{}^c = 2i\gamma^c{}_{\alpha\beta} + 2K^c \gamma^3{}_{\alpha\beta}, \qquad F_{\alpha\beta} = 2S\gamma^3{}_{\alpha\beta}. \tag{2.157}$$

Whereas the scalar superfield S was already present in the original supergravity model of Howe where the bosonic curvature r was part of the θ^2-component of S (cf. (2.120) and (2.123)) the vector superfield K^c enters as a new superfield playing a similar role for the bosonic torsion τ^c.

It should be stressed that the superspace constraints do not break the symmetries of the theory such as superdiffeomorphisms or Lorentz supertransformations. The constraints merely reduce the number of independent components of the supervielbein $E_A{}^M$ and the Lorentz superconnection Ω_A as can be seen in Section 2.5.2 below.

2.5.1 Bianchi Identities for New Constraints

The Bianchi identities $\Delta_{ABC}{}^D = R_{[ABC]}{}^D$ (cf. (2.55)) and $\Delta_{ABC} = 0$ (cf. (2.68)) are relations between the components of supertorsion and supercurvature and their derivatives. In the presence of constraints they can be used to determine an independent set of superfields from which all components of supertorsion and supercurvature can be derived. In our case this set turns out to be formed by the superfields S and K^c of ansatz (2.157).

The decompositions similar to the one of the Rarita-Schwinger field (B.48) of Appendix B.2 are very useful for the derivation of the formulae below. For present needs we employ

$$F_{a\beta} = F^{-}{}_{a\beta} - \gamma_{a\beta}{}^{\alpha} F^{+}{}_{\alpha}, \tag{2.158}$$
$$T_{a\beta}{}^{c} = T^{-}{}_{a\beta}{}^{c} - \gamma_{a\beta}{}^{\alpha} T^{+}{}_{\alpha}{}^{c}, \tag{2.159}$$
$$T_{a\beta}{}^{\gamma} = T^{-}{}_{a\beta}{}^{\gamma} - \gamma_{a\beta}{}^{\alpha} T^{+}{}_{\alpha}{}^{\gamma}, \tag{2.160}$$

where F^- and T^- are γ^a traceless, e.g. we have $\gamma^a{}_\alpha{}^\beta F^{-}{}_{a\beta} = 0$ in conformance with (B.50). Further helpful formulae to derive the results below are again found in Appendix B.2 in particular the relations (B.62)–(B.69).

The Bianchi identity $\Delta_{\alpha\beta\gamma} = 0$ yields, for the parts of $F_{a\beta}$ as defined by (2.158),

$$F^{-}{}_{a\beta} = 0, \qquad \nabla_\alpha S = -iK_\alpha{}^\beta F^{+}{}_\beta. \tag{2.161}$$

Here the superfield operator

$$K_\alpha{}^\beta := \gamma^3{}_\alpha{}^\beta + iK^a \gamma_{a\alpha}{}^\beta \tag{2.162}$$

was introduced because this particular shift of γ^3 by the vector superfield K^a is encountered frequently. Due to

$$K_\alpha{}^\beta K_\beta{}^\gamma = (1 - K^a K_a)\, \delta_\alpha{}^\gamma \tag{2.163}$$

$K_\alpha{}^\beta$ can be inverted if body$(K^a K_a) \neq 1$:

$$K^{-1}{}_\alpha{}^\beta = (1 - K^a K_a)^{-1} K_\alpha{}^\beta. \tag{2.164}$$

Therefore, from (2.161)

$$F_{a\beta} = -(\gamma_a F^+)_\beta, \qquad F^{+}{}_\alpha = iK^{-1}{}_\alpha{}^\beta \nabla_\beta S \tag{2.165}$$

is obtained.

In the same way the Bianchi identity $\Delta_{\alpha\beta\gamma}{}^d = R_{\alpha\beta\gamma}{}^d$ yields for the separate parts of the decomposition of $T_{a\beta}{}^c$ (cf. (2.159))

$$T^{-}{}_{a\beta}{}^c = 0, \qquad \nabla_\alpha K^c = -iK_\alpha{}^\beta T^{+}{}_\beta{}^c. \tag{2.166}$$

From $\Delta_{\alpha\beta\gamma}{}^\delta = R_{\alpha\beta\gamma}{}^\delta$ we get for $T_{a\beta}{}^\gamma$ in its decomposition (2.160)

$$T^{-}{}_{a\beta}{}^\gamma = 0, \qquad K_\beta{}^\alpha T^{+}{}_\alpha{}^\gamma = \frac{i}{2} S \gamma^3{}_\beta{}^\gamma. \tag{2.167}$$

The second equation of (2.167) can be rewritten using a rescaled superfield S'

$$S = (1 - K^a K_a) S', \qquad T^{+}{}_\alpha{}^\beta = \frac{i}{2} S' (K\gamma^3)_\alpha{}^\beta, \tag{2.168}$$

where S' can be easily calculated from $S' = -iT^{+}{}_{\alpha}{}^{\alpha}$.

Further relations are derived from the Bianchi identity $\Delta_{a\beta\gamma} = 0$. The symmetrized contraction $\gamma_{(b}{}^{\gamma\beta}\Delta_{a)\beta\gamma} = 0$ yields

$$\epsilon^{\beta\alpha}\nabla_{\alpha}F^{+}{}_{\beta} = (T^{+a}\gamma_a F^{+}), \tag{2.169}$$

and the antisymmetric part $\gamma_{[b}{}^{\gamma\beta}\Delta_{a]\beta\gamma} = 0$ gives

$$\gamma^{3\beta\alpha}\nabla_{\alpha}F^{+}{}_{\beta} = -(T^{+a}\gamma_a\gamma^3F^{+}) - 2iS'S + i\epsilon^{ba}F_{ab}. \tag{2.170}$$

The contraction $\gamma^{3\gamma\beta}\Delta_{a\beta\gamma} = 0$ leads to the same relations (2.169) and (2.170), but needs more tedious calculations.

Similar formulae are derived from the Bianchi identity $\Delta_{\alpha\beta c}{}^{d} = R_{[\alpha\beta c]}{}^{d}$. From the symmetrized contraction $\gamma_{(b|}{}^{\alpha\beta}\Delta_{\alpha\beta|c)}{}^{d}$ one gets the relation

$$\epsilon^{\beta\alpha}\nabla_{\alpha}T^{+}{}_{\beta}{}^{d} = (T^{+a}\gamma_a T^{+d}) + 2iS'K^a\epsilon_a{}^d, \tag{2.171}$$

and from the antisymmetrized contraction $\gamma_{[b|}{}^{\alpha\beta}\Delta_{\alpha\beta|c]}{}^{d}$

$$\gamma^{3\beta\alpha}\nabla_{\alpha}T^{+}{}_{\beta}{}^{d} = -(T^{+a}\gamma_a\gamma^3T^{+d}) - 2iS'K^d + i\epsilon^{ba}T_{ab}{}^{d}. \tag{2.172}$$

From the Bianchi identity $\Delta_{\alpha\beta c}{}^{\delta} = R_{[\alpha\beta c]}{}^{\delta}$ also an expression for $T_{ab}{}^{\gamma}$ could be obtained.

Finally we summarize the expressions for the supercurvature and supertorsion components in terms of the independent and unconstrained superfields S and K^a; the latter is also used to build the matrix K and its inverse K^{-1} (cf. (2.162) and (2.164)):

$$F_{\alpha\beta} = 2S\gamma^3{}_{\alpha\beta}, \qquad T_{\alpha\beta}{}^{c} = 2i\gamma^c{}_{\alpha\beta} + 2K^c\gamma^3{}_{\alpha\beta}, \tag{2.173}$$

$$F_{a\beta} = -i(\gamma_a K^{-1})_{\beta}{}^{\alpha}(\nabla_{\alpha}S), \qquad T_{a\beta}{}^{c} = -i(\gamma_a K^{-1})_{\beta}{}^{\alpha}(\nabla_{\alpha}K^c), \tag{2.174}$$

$$T_{a\beta}{}^{\gamma} = -\frac{i}{2}S(\gamma_a K^{-1}\gamma^3)_{\beta}{}^{\gamma}, \tag{2.175}$$

and the more complicated expressions

$$\begin{aligned}\frac{1}{2}\epsilon^{ba}F_{ab} &= \frac{1}{2}\gamma^{3\beta\alpha}\nabla_{\alpha}(K^{-1}{}_{\beta}{}^{\gamma}\nabla_{\gamma}S)\\ &\quad + \frac{i}{2}(K^{-1}\gamma_a\gamma^3K^{-1})^{\beta\alpha}(\nabla_{\alpha}K^a)(\nabla_{\beta}S) + \frac{S^2}{1-K^aK_a},\end{aligned} \tag{2.176}$$

$$\begin{aligned}\frac{1}{2}\epsilon^{ba}T_{ab}{}^{c} &= \frac{1}{2}\gamma^{3\beta\alpha}\nabla_{\alpha}(K^{-1}{}_{\beta}{}^{\gamma}\nabla_{\gamma}K^c)\\ &\quad + \frac{i}{2}(K^{-1}\gamma_a\gamma^3K^{-1})^{\beta\alpha}(\nabla_{\alpha}K^a)(\nabla_{\beta}K^c) + \frac{S}{1-K^aK_a}K^c.\end{aligned} \tag{2.177}$$

2.5.2 Solution of the Constraints

The solution of the new supergravity constraints (2.156) can be derived solely with superfield methods. To achieve that goal the formulae for the supertorsion components (2.44)–(2.49) and for the supercurvature components (2.58)–(2.65), in each case consequences of the restricted tangent space group (2.36), are employed. However, the decomposition of the supervielbein in terms of the superfields ${B_m}^a$, ${\Psi_m}^a$, ${\Phi_\mu}^a$ and ${B_\mu}^\alpha$ (cf. (2.69)) and the ensuing formulae for the anholonomicity coefficients (2.75)–(2.80) turn out to be essential. The method for solving the supergravity constraints in terms of superfields employed here is similar to the one developed in [140].

We start with the ansatz (2.157) for ${T_{\alpha\beta}}^c$. Contraction with ${\gamma_a}^{\beta\alpha}$ leads to the second constraint ${\gamma_a}^{\beta\alpha}{T_{\alpha\beta}}^c = -4i{\delta_a}^c$ (cf. (2.156)). Looking into the expression for the supertorsion (2.45) we recognize that the superconnection drops out there. The remaining anholonomicity coefficient, calculated according to (2.75), allows to express ${B_a}^n$ in terms of ${B_\alpha}^\mu$ and ${\Phi_\nu}^n$:

$$ {B_a}^n = -\frac{i}{2}{\gamma_a}^{\beta\alpha}{B_\alpha}^\mu {B_\beta}^\nu \left({\Phi_\mu}^l \partial_l {\Phi_\nu}^n - \partial_\mu {\Phi_\nu}^n\right). \tag{2.178} $$

The contraction with $\gamma^{3\beta\alpha}$ of ${T_{\alpha\beta}}^c$ (cf. (2.157)) yields $K^c = -\frac{1}{4}\gamma^{3\beta\alpha}{T_{\alpha\beta}}^c$. Again using (2.45) and (2.75) together with $K^n := K^c {B_c}^n$

$$ K^n = \frac{1}{2}\gamma^{3\beta\alpha}{B_\alpha}^\mu {B_\beta}^\nu ({\Phi_\mu}^l \partial_l {\Phi_\nu}^n - \partial_\mu {\Phi_\nu}^n) \tag{2.179} $$

is obtained, thus expressing K^n in terms of ${B_\alpha}^\mu$ and ${\Phi_\nu}^n$.

Next we turn to the investigation of the constraint ${T_{\alpha\beta}}^\gamma = 0$. Contracting (2.44) with ${\gamma_a}^{\beta\alpha}$ and $\gamma^{3\beta\alpha}$ yields

$$ -{\gamma_a}^{\beta\alpha}{C_{\alpha\beta}}^\gamma + \Omega^\alpha (\gamma_a\gamma^3)_\alpha{}^\gamma = 0, \tag{2.180} $$

$$ -\gamma^{3\beta\alpha}{C_{\alpha\beta}}^\gamma + \Omega^\gamma = 0. \tag{2.181} $$

These two equations can be used to eliminate ${\Psi_a}^\nu$ and Ω^α: Indeed, using formula (2.76) to calculate the anholonomicity coefficients and observing that parts of that formula are of the form (2.178) and (2.179) we derive

$$ {\Psi_a}^\nu = -\frac{i}{2}{\gamma_a}^{\beta\alpha}{B_\alpha}^\mu \left({\Phi_\mu}^l \partial_l {B_\beta}^\nu - \partial_\mu {B_\beta}^\nu\right) - \frac{i}{4}\Omega^\alpha (\gamma_a\gamma^3)_\alpha{}^\beta {B_\beta}^\nu, \tag{2.182} $$

$$ -4K^n {\Psi_n}^\gamma + 2\gamma^{3\beta\alpha}{B_\alpha}^\mu \left({\Phi_\mu}^m \partial_m {B_\beta}^\lambda - \partial_\mu {B_\beta}^\lambda\right) {B_\lambda}^\gamma + \Omega^\gamma = 0. \tag{2.183} $$

The first equation expresses ${\Psi_a}^\nu$ in terms of ${B_\alpha}^\mu$, ${\Phi_\nu}^n$ and Ω^α, whereas an appropriate combination of both equations gives

$$ (\Omega K \gamma^3)^\gamma = -2K^{\beta\alpha}{B_\alpha}^\mu \left({\Phi_\mu}^l \partial_l {B_\beta}^\nu - \partial_\mu {B_\beta}^\nu\right) {B_\nu}^\gamma, \tag{2.184} $$

so that Ω^α can be eliminated in favour of $B_\alpha{}^\mu$, $\Phi_\nu{}^n$ and K^c:

$$\Omega^\delta = -2K^{\beta\alpha}B_\alpha{}^\mu\left(\Phi_\mu{}^l\partial_l B_\beta{}^\nu - \partial_\mu B_\beta{}^\nu\right)B_\nu{}^\gamma(\gamma^3K^{-1})_\gamma{}^\delta. \tag{2.185}$$

We recall that $B_\nu{}^\gamma$ is the inverse of $B_\alpha{}^\mu$ defined in (2.71) and that the matrix $K_\beta{}^\alpha$ and its inverse were given in (2.162) and (2.164).

The third constraint $\gamma_a{}^{\beta\alpha}F_{\alpha\beta} = 0$ of (2.156) allows to eliminate the superconnection component Ω_a. Calculating the supercurvature according to (2.52) the expression

$$2\gamma_a{}^{\beta\alpha}(\nabla_\alpha\Omega_\beta) + \gamma_a{}^{\beta\alpha}T_{\alpha\beta}{}^\gamma\Omega_\gamma + \gamma_a{}^{\beta\alpha}T_{\alpha\beta}{}^c\Omega_c = 0 \tag{2.186}$$

is found. Then the constraints on the supertorsion (cf. (2.156)) immediately lead to

$$\Omega_a = -\frac{i}{2}\gamma_a{}^{\beta\alpha}(\nabla_\alpha\Omega_\beta) = \frac{i}{2}\gamma_a{}^{\beta\alpha}B_\alpha{}^\mu\left(\Phi_\mu{}^l\partial_l\Omega_\beta - \partial_\mu\Omega_\beta\right). \tag{2.187}$$

This gives a complete solution of the new supergravity constraints (2.156) in terms of the unconstrained and independent superfields $B_\alpha{}^\mu$ and $\Phi_\nu{}^n$. Due to the supergravity constraints the superfields $B_a{}^n$ (2.178) and $\Psi_a{}^\nu$ (2.182), member of the supervielbein, as well as the whole superconnection Ω_α (2.185) and Ω_a (2.187) were eliminated. The vector superfield K^a appeared as the special combination (2.179).

2.5.3 Physical Fields and Gauge Fixing

Although the superfield solution given in the previous section looks quite pleasant, calculating the θ-expansion to recover the physical x-space fields is cumbersome. Let us have a quick look at the number of components that remain: The $6+4+2 = 12$ superspace constraints (2.156) reduce the original $16+4 = 20$ superfields formed by $E_M{}^A$ and Ω_M down to $4+4 = 8$ superfields contained in $B_\alpha{}^\mu$ and $\Phi_\nu{}^n$. Thus, the remaining number of x-space fields is $4 \times 8 = 32$. This is still more than the $4+4+2 = 10$ components of the zweibein $e_m{}^a$, the Rarita-Schwinger field $\psi_m{}^\alpha$ and the Lorentz connection ω_m which we want to accommodate within the superfields.

An additional complication is that the identification of the physical x-space fields is not made within the independent superfields $B_\alpha{}^\mu$ and $\Phi_\nu{}^n$, but with respect to the already eliminated superfields to lowest order in θ. The identification is expected to be

$$E_m{}^a| = e_m{}^a, \qquad E_m{}^\alpha| = \psi_m{}^\alpha, \qquad \Omega_m| = \omega_m. \tag{2.188}$$

Whereas the first two relations will be found to hold, the identification of ω_m in the explicit formulae below will be different. It will be made at first order in θ of (2.222) leading to the result (2.223) for Ω_m. A simple redefinition of ω_m will fix that disagreement.

In order to reduce the 32 independent x-space fields a Wess-Zumino type gauge fixing is chosen. It is the same as in the original model of Howe [46]. In particular to zeroth order in θ there are the $4+4+2=10$ conditions

$$E_\mu{}^\alpha| = \delta_\mu{}^\alpha, \qquad E_\mu{}^a = 0|, \qquad \Omega_\mu| = 0 \tag{2.189}$$

and to first order the $2+2+1=5$ conditions

$$\partial_{[\nu} E_{\mu]}{}^\alpha| = 0, \qquad \partial_{[\nu} E_{\mu]}{}^a| = 0, \qquad \partial_{[\nu} \Omega_{\mu]}| = 0, \tag{2.190}$$

which reduce the number of independent component fields down to 17. A corresponding reduction also follows for the number of symmetries contained within superdiffeomorphism and Lorentz supertransformation. The same considerations as in Section 2.3.2 apply, for a detailed analysis cf. [141].

The remaining 17 degrees of freedom are constituted by the 10 components of zweibein, Rarita-Schwinger field and Lorentz connection, and by additional 7 components consisting of the well-known auxiliary field A and the newly introduced fields k^a and $\varphi_m{}^\alpha$. The identification for k^a is

$$K^a| = k^a, \tag{2.191}$$

the others at the end of the calculation are to be identified with zeroth components of supertorsion and supercurvature.

2.5.4 Component Fields of New Supergravity

To recover the component fields the superfield expressions (2.178), (2.179), (2.182), (2.183) and (2.187) have to be worked out order by order in the anticommutative coordinate θ. For the decomposition of spin-tensors we refer to Appendix B.2. The reader should especially consult (B.48) where the decomposition of the Rarita-Schwinger field is given.

In the calculations new Γ-matrices, dependent on the vector field k^a (cf. (2.191)), were encountered frequently:

$$\Gamma^a := \gamma^a - ik^a\gamma^3, \qquad \Gamma^3 := \gamma^3 + ik^a\gamma_a. \tag{2.192}$$

Their (anti-)commutator algebra and other properties as well as formulae used in the calculations can be found in Appendix B.3.

In the θ-expansion some covariant derivatives of x-space fields are encountered. These are, respectively, the torsion, the curvature, the covariant

derivative of the Rarita-Schwinger field

$$\begin{aligned}
t_{ab}{}^{c} &= -c_{ab}{}^{c} + \omega_a \epsilon_b{}^{c} - \omega_b \epsilon_a{}^{c} - Ak_a \epsilon_b{}^{c} + Ak_b \epsilon_a{}^{c} \\
&\quad - 2i(\psi_a \Gamma^c \psi_b) - 4i\epsilon_{ab}(\psi\gamma^3\varphi^c), \qquad (2.193) \\
f_{mn} &= \partial_m \omega_n - \partial_n \omega_m - \partial_m(Ak_n) + \partial_n(Ak_m) \\
&\quad - 2(1-k^2)A(\psi_m \gamma^3 \psi_n) - 4i\epsilon_{mn}(\psi\gamma^3\Gamma^3\gamma^3\sigma) \\
&\quad - 4A\epsilon_{mn}\epsilon^{dc}(\psi\gamma^3\Gamma_c\varphi_d) + 4iA\epsilon_{mn}(\psi\gamma^3\varphi^c)k_c, \qquad (2.194) \\
\sigma_{mn}{}^{\gamma} &= \partial_m \psi_n{}^{\gamma} - \frac{1}{2}\omega_m(\psi_n\gamma^3)^\gamma - \frac{i}{2}A\epsilon_m{}^{l}(\psi_n\Gamma_l\gamma^3)^\gamma - (m \leftrightarrow n), \qquad (2.195)
\end{aligned}$$

and the covariant derivative of k^b

$$\mathcal{D}_a k^b = \partial_a k^b + \omega_a k^c \epsilon_c{}^{b} - Ak_a k^c \epsilon_c{}^{b} - 2(\psi_a \Gamma^3 \varphi^b). \qquad (2.196)$$

They are (obviously) covariant with respect to Lorentz transformations, but they are also covariant with respect to supersymmetry transformations. For the Hodge duals $\tau^c = \frac{1}{2}\epsilon^{ba} t_{ab}{}^{c}$ and $\sigma^\gamma = \frac{1}{2}\epsilon^{nm}\sigma_{mn}{}^{\gamma}$ one obtains (cf. (A.13) and also (A.17))

$$\begin{aligned}
\tau^c &= \tilde{\omega}^c - \omega^c + Ak^c - i\epsilon^{ba}(\psi_a\Gamma^c\psi_b) - 4i(\psi\gamma^3\varphi^c), \qquad (2.197) \\
\sigma^\gamma &= \epsilon^{nm}\partial_m\psi_n{}^{\gamma} - \frac{1}{2}\epsilon^{nm}\omega_m(\psi_n\gamma^3)^\gamma - \frac{i}{2}A(\psi_n\Gamma^n\gamma^3)^\gamma. \qquad (2.198)
\end{aligned}$$

Here it becomes obvious that the identification of ω_m in (2.223) was not the best choice. This could be fixed by the replacement $\omega_a \to \omega'_a = \omega_a + Ak_a$.

First we summarize the results for the decomposition superfields of the supervielbein. For $B_a{}^{m}$ and its inverse $B_m{}^{a}$ (cf. (2.71)), where the zweibein $e_a{}^{m}$ and $e_m{}^{a}$ are the zeroth components, we obtain

$$\begin{aligned}
B_a{}^{m} &= e_a{}^{m} - 2i(\theta\Gamma^m\psi_a) - 2i(\theta\gamma_a\varphi^m) \\
&\quad + \frac{1}{2}\theta^2\left[-\epsilon_a{}^{b}(\mathcal{D}_b k^c)e_c{}^{m} + k_a\tau^m - \omega_a k^m - Ak_a k^m - Ae_a{}^{m} - k^2 Ae_a{}^{m}\right] \\
&\quad + \frac{1}{2}\theta^2\left[-4(\psi_a\Gamma^b\Gamma^m\psi_b) - 6(\psi_a\varphi^m) - 4(\psi_b\Gamma^m\gamma_a\varphi^b)\right], \qquad (2.199) \\
B_m{}^{a} &= e_m{}^{a} + 2i(\theta\Gamma^a\psi_m) + 2i(\theta\gamma_m\varphi^a) \\
&\quad + \frac{1}{2}\theta^2\left[\epsilon_m{}^{n}(\mathcal{D}_n k^a) - k_m\tau^a + \omega_m k^a + Ak_m k^a + Ae_m{}^{a} + k^2 Ae_m{}^{a}\right] \\
&\quad + \frac{1}{2}\theta^2\left[-2(\psi_m\varphi^a) + 4ik^b(\psi_m\gamma^3\gamma_b\varphi^a) - 4(\varphi^b\gamma_m\gamma_b\varphi^a)\right]. \qquad (2.200)
\end{aligned}$$

The component fields in $B_\alpha{}^\mu$ and its inverse $B_\mu{}^\alpha$ (cf. again (2.71)) are

$$\begin{aligned} B_\alpha{}^\mu &= \delta_\alpha{}^\mu + i(\theta\Gamma^b)_\alpha \psi_b{}^\mu \\ &\quad + \frac{1}{2}\theta^2\left[-\frac{i}{2}\omega_b(\Gamma^b\gamma^3)_\alpha{}^\mu - \frac{1}{2}A(\Gamma^3\gamma^3)_\alpha{}^\mu + (\Gamma^b\Gamma^c\psi_b)_\alpha\psi_c{}^\mu + 4\varphi^c{}_\alpha\psi_c{}^\mu\right], \end{aligned} \tag{2.201}$$

$$\begin{aligned} B_\mu{}^\alpha &= \delta_\mu{}^\alpha - i(\theta\Gamma^b)_\mu \psi_b{}^\alpha \\ &\quad + \frac{1}{2}\theta^2\left[\frac{i}{2}\omega_b(\Gamma^b\gamma^3)_\mu{}^\alpha + \frac{1}{2}A(\Gamma^3\gamma^3)_\mu{}^\alpha - 2(\Gamma^b\Gamma^c\psi_b)_\mu\psi_c{}^\alpha - 4\varphi^c{}_\mu\psi_c{}^\alpha\right]. \end{aligned} \tag{2.202}$$

The various contractions with $\Phi_\mu{}^m$ (cf. (2.72)) are given by

$$\Phi_\mu{}^m = i(\theta\Gamma^m)_\mu + \frac{1}{2}\theta^2\left[4\varphi^m{}_\mu + 2(\Gamma^b\Gamma^m\psi_b)_\mu\right], \tag{2.203}$$

$$\Phi_\mu{}^a = i(\theta\Gamma^a)_\mu + \frac{1}{2}\theta^2\left[2ik^b(\gamma^3\gamma_b\varphi^a)_\mu\right], \tag{2.204}$$

$$\Phi_\alpha{}^a = i(\theta\Gamma^a)_\alpha + \frac{1}{2}\theta^2\left[2ik^b(\gamma^3\gamma_b\varphi^a)_\alpha - (\Gamma^b\Gamma^a\psi_b)_\alpha\right], \tag{2.205}$$

$$\Phi_\alpha{}^m = i(\theta\Gamma^m)_\alpha + \frac{1}{2}\theta^2\left[4\varphi^m{}_\alpha + (\Gamma^b\Gamma^m\psi_b)_\alpha\right]. \tag{2.206}$$

The superfield $\Psi_m{}^\alpha$ with the Rarita-Schwinger $\psi_m{}^\alpha$ field at lowest order and its various contractions (cf. (2.73)) read

$$\begin{aligned} \Psi_m{}^\alpha &= \psi_m{}^\alpha - \frac{1}{2}\omega_m(\theta\gamma^3)^\alpha - \frac{i}{2}A\epsilon_m{}^n(\theta\Gamma_n\gamma^3)^\alpha \\ &\quad + \frac{1}{2}\theta^2\left[-\frac{3}{2}A(\psi_m\Gamma^3\gamma^3)^\alpha - 2A\epsilon_m{}^n(\varphi_n\gamma^3)^\alpha - i\sigma_{mn}{}^\beta(\gamma^3\Gamma^n\gamma^3)_\beta{}^\alpha\right], \end{aligned} \tag{2.207}$$

$$\Psi_m{}^\mu = \psi_m{}^\mu - \frac{1}{2}\omega_m(\theta\gamma^3)^\mu - \frac{i}{2}A\epsilon_m{}^n(\theta\Gamma_n\gamma^3)^\mu + i(\theta\Gamma^n\psi_m)\psi_n{}^\mu + \frac{1}{2}\theta^2\left[\cdots\right], \tag{2.208}$$

$$\begin{aligned} \Psi_a{}^\alpha &= \psi_a{}^\alpha - \frac{1}{2}\omega_a(\theta\gamma^3)^\alpha - \frac{i}{2}A\epsilon_a{}^b(\theta\Gamma_b\gamma^3)^\alpha - 2i(\theta\Gamma^b\psi_a)\psi_b{}^\alpha - 2i(\theta\gamma_a\varphi^b)\psi_b{}^\alpha \\ &\quad + \frac{1}{2}\theta^2\left[-\epsilon_a{}^b(\mathcal{D}_b k^c)\psi_c{}^\alpha + k_a\tau^b\psi_b{}^\alpha - i\sigma_{ab}{}^\beta(\gamma^3\Gamma^b\gamma^3)_\beta{}^\alpha\right] \\ &\quad + \frac{1}{2}\theta^2\left[i\omega_b(\psi_a\Gamma^b\gamma^3)^\alpha - \omega_a k^b\psi_b{}^\alpha + i\omega_b(\varphi^b\gamma_a\gamma^3)^\alpha\right] \\ &\quad + \frac{1}{2}\theta^2\left[-\frac{1}{2}A(\psi_a\gamma^3\Gamma^3)^\alpha - Ak_a k^b\psi_b{}^\alpha - k^2 A\psi_a{}^\alpha - A\epsilon^{cb}(\varphi_b\Gamma_c\gamma_a\gamma^3)^\alpha\right] \\ &\quad\quad + \frac{1}{2}\theta^2\left[-4(\psi_a\Gamma^b\Gamma^c\psi_b)\psi_c{}^\alpha - 6(\psi_a\varphi^b)\psi_b{}^\alpha - 4(\psi_b\Gamma^c\gamma_a\varphi^b)\psi_c{}^\alpha\right], \end{aligned} \tag{2.209}$$

$$\begin{aligned}
\Psi_a{}^\mu = {}& \psi_a{}^\mu - \frac{1}{2}\omega_a(\theta\gamma^3)^\mu - \frac{i}{2}A\epsilon_a{}^b(\theta\Gamma_b\gamma^3)^\mu - i(\theta\Gamma^b\psi_a)\psi_b{}^\mu - 2i(\theta\gamma_a\varphi^b)\psi_b{}^\mu \\
& + \frac{1}{2}\theta^2\left[-\epsilon_a{}^b(\mathcal{D}_b k^c)\psi_c{}^\mu + k_a\tau^b\psi_b{}^\mu - i\sigma_{ab}{}^\beta(\gamma^3\Gamma^b\gamma^3)_\beta{}^\mu\right] \\
& + \frac{1}{2}\theta^2\left[\frac{i}{2}\omega_b(\psi_a\Gamma^b\gamma^3)^\mu + i\omega_b(\varphi^b\gamma_a\gamma^3)^\mu\right] \\
& + \frac{1}{2}\theta^2\left[-Ak_a k^b\psi_b{}^\mu - Ak^2\psi_a{}^\mu - A\epsilon^{cb}(\varphi_b\Gamma_c\gamma_a\gamma^3)^\mu\right] \\
& \quad + \frac{1}{2}\theta^2\left[-(\psi_a\Gamma^b\Gamma^c\psi_b)\psi_c{}^\mu - 2(\psi_a\varphi^b)\psi_b{}^\mu - 2(\psi_b\Gamma^c\gamma_a\varphi^b)\psi_c{}^\mu\right].
\end{aligned} \tag{2.210}$$

The K^a vector superfield constituting a new multiplet of fields for supergravity with torsion and the contraction $K^m = K^a B_a{}^m$ are given by

$$\begin{aligned}
K^a = {}& k^a + 2(\theta\Gamma^3\varphi^a) \\
& \quad + \frac{1}{2}\theta^2\left[\left(1-k^2\right)\left(\tau^a - 2Ak^a\right) + k^c\epsilon_c{}^b(\mathcal{D}_b k^a) + 4i(\varphi^b\Gamma^3\gamma_b\varphi^a)\right],
\end{aligned} \tag{2.211}$$

$$\begin{aligned}
K^m = {}& k^m + 2(\theta\gamma^3\varphi^m) - 2ik^b(\theta\Gamma^m\psi_b) \\
& + \frac{1}{2}\theta^2\left[\tau^m - k^b\omega_b k^m - 3Ak^m - 4k^c(\psi_c\Gamma^b\Gamma^m\psi_b)\right] \\
& \quad + \frac{1}{2}\theta^2\left[-6k^c(\psi_c\varphi^m) - 4i(\psi_b\Gamma^m\gamma^3\varphi^b)\right].
\end{aligned} \tag{2.212}$$

Here the vector k^a appears at zeroth order (as was already mentioned in (2.191)), the spin-vector $\varphi_m{}^\alpha$ at first order and the torsion τ^a at second order in θ, constituting a new multiplet of x-space fields $\{k^a, \varphi_m{}^\alpha, \tau^a\}$, where the torsion τ^a could also be replaced by the Lorentz connection ω_a.

The inverse supervielbein $E_A{}^M$ can be derived from the above decom-

position superfields according to (2.70)

$$\begin{aligned}
E_a{}^m &= e_a{}^m - i(\theta\Gamma^m\psi_a) - 2i(\theta\gamma_a\varphi^m) \\
&\quad + \frac{1}{2}\theta^2\left[-\epsilon_a{}^b(\mathcal{D}_b k^c)e_c{}^m + k_a\tau^m - Ak_a k^m - Ak^2 e_a{}^m\right] \\
&\quad\quad + \frac{1}{2}\theta^2\left[-(\psi_a\Gamma^n\Gamma^m\psi_n) - 2(\psi_a\varphi^m) - 2(\psi_b\Gamma^m\gamma_a\varphi^b)\right],
\end{aligned} \tag{2.213}$$

$$\begin{aligned}
E_a{}^\mu &= -\psi_a{}^\mu + \frac{1}{2}\omega_a(\theta\gamma^3)^\mu + \frac{i}{2}A\epsilon_a{}^b(\theta\Gamma_b\gamma^3)^\mu + i(\theta\Gamma^b\psi_a)\psi_b{}^\mu + 2i(\theta\gamma_a\varphi^b)\psi_b{}^\mu \\
&\quad + \frac{1}{2}\theta^2\left[\epsilon_a{}^b(\mathcal{D}_b k^c)\psi_c{}^\mu - k_a\tau^b\psi_b{}^\mu + i\sigma_{ab}{}^\beta(\gamma^3\Gamma^b\gamma^3)_\beta{}^\mu\right] \\
&\quad + \frac{1}{2}\theta^2\left[-\frac{i}{2}\omega_b(\psi_a\Gamma^b\gamma^3)^\mu - i\omega_b(\varphi^b\gamma_a\gamma^3)^\mu\right] \\
&\quad + \frac{1}{2}\theta^2\left[Ak_a k^b\psi_b{}^\mu + Ak^2\psi_a{}^\mu + A\epsilon^{cb}(\varphi_b\Gamma_c\gamma_a\gamma^3)^\mu\right] \\
&\quad\quad + \frac{1}{2}\theta^2\left[(\psi_a\Gamma^b\Gamma^c\psi_b)\psi_c{}^\mu + 2(\psi_a\varphi^b)\psi_b{}^\mu + 2(\psi_b\Gamma^c\gamma_a\varphi^b)\psi_c{}^\mu\right],
\end{aligned} \tag{2.214}$$

$$E_\alpha{}^m = -i(\theta\Gamma^m)_\alpha + \frac{1}{2}\theta^2\left[-4\varphi^m{}_\alpha - (\Gamma^b\Gamma^m\psi_b)_\alpha\right], \tag{2.215}$$

$$\begin{aligned}
E_\alpha{}^\mu &= \delta_\alpha{}^\mu + i(\theta\Gamma^b)_\alpha\psi_b{}^\mu \\
&\quad + \frac{1}{2}\theta^2\left[-\frac{i}{2}\omega_b(\Gamma^b\gamma^3)_\alpha{}^\mu - \frac{1}{2}A(\Gamma^3\gamma^3)_\alpha{}^\mu + (\Gamma^b\Gamma^c\psi_b)_\alpha\psi_c{}^\mu + 4\varphi^c{}_\alpha\psi_c{}^\mu\right].
\end{aligned} \tag{2.216}$$

For the θ-expansion of the supervielbein $E_M{}^A$ now applying the decomposition (2.69) we arrive at

$$\begin{aligned}
E_m{}^a &= e_m{}^a + 2i(\theta\Gamma^a\psi_m) + 2i(\theta\gamma_m\varphi^a) \\
&\quad + \frac{1}{2}\theta^2\left[\epsilon_m{}^n(\mathcal{D}_n k^a) - k_m\tau^a + \omega_m k^a + Ak_m k^a + Ae_m{}^a + k^2 Ae_m{}^a\right] \\
&\quad\quad + \frac{1}{2}\theta^2\left[-2(\psi_m\varphi^a) + 4ik^b(\psi_m\gamma^3\gamma_b\varphi^a) - 4(\varphi^b\gamma_m\gamma_b\varphi^a)\right],
\end{aligned} \tag{2.217}$$

$$\begin{aligned}
E_m{}^\alpha &= \psi_m{}^\alpha - \frac{1}{2}\omega_m(\theta\gamma^3)^\alpha - \frac{i}{2}A\epsilon_m{}^n(\theta\Gamma_n\gamma^3)^\alpha \\
&\quad + \frac{1}{2}\theta^2\left[-\frac{3}{2}A(\psi_m\Gamma^3\gamma^3)^\alpha - 2A\epsilon_m{}^n(\varphi_n\gamma^3)^\alpha - i\sigma_{mn}{}^\beta(\gamma^3\Gamma^n\gamma^3)_\beta{}^\alpha\right],
\end{aligned} \tag{2.218}$$

$$E_\mu{}^a = i(\theta\Gamma^a)_\mu + \frac{1}{2}\theta^2\left[2ik^b(\gamma^3\gamma_b\varphi^a)_\mu\right], \tag{2.219}$$

$$E_\mu{}^\alpha = \delta_\mu{}^\alpha + \frac{1}{2}\theta^2\left[-\frac{1}{2}A(\Gamma^3\gamma^3)_\mu{}^\alpha\right]. \tag{2.220}$$

Finally the Lorentz superconnection Ω_A reads

$$\Omega_a = \omega_a - Ak_a - i(\theta\Gamma^b\psi_a)\omega_b + A(\theta\Gamma^3\psi_a) - 2i(\theta\gamma_a\varphi^b)\omega_b + 2A\epsilon^{cb}(\theta\gamma_a\Gamma_b\varphi_c) + 2i(\theta\gamma_a\Gamma^3\gamma^3\sigma) + \frac{1}{2}\theta^2[\cdots], \tag{2.221}$$

$$\Omega_\alpha = -i(\theta\Gamma^b)_\alpha\omega_b + A(\theta\Gamma^3)_\alpha + \frac{1}{2}\theta^2\left[-(\Gamma^c\Gamma^b\psi_c)_\alpha\omega_b + iA(\Gamma^3\Gamma^b\psi_b)_\alpha\right] + \frac{1}{2}\theta^2\left[-4\varphi^b{}_\alpha\omega_b - 4iA\epsilon^{cb}(\Gamma_b\varphi_c)_\alpha + 4(\Gamma^3\gamma^3\sigma)_\alpha\right], \tag{2.222}$$

and for $\Omega_M = E_M{}^A\Omega_A$ we obtain

$$\Omega_m = \omega_m - Ak_m + 2(1-k^2)A(\theta\gamma^3\psi_m) + 2A\epsilon^{cb}(\theta\gamma_m\Gamma_b\varphi_c) - 2iA(\theta\gamma_m\varphi^b)k_b + 2i(\theta\gamma_m\Gamma^3\gamma^3\sigma) + \frac{1}{2}[\cdots], \tag{2.223}$$

$$\Omega_\mu = (1-k^2)A(\theta\gamma^3)_\mu + \frac{1}{2}\theta^2[\cdots]. \tag{2.224}$$

When terms were omitted in the formulae above there is no difficulty of principle to work them out, but it is tedious to do so.

2.5.5 Supertorsion and Supercurvature

Finally we calculate the supertorsion and supercurvature components to zeroth order in θ. There the auxiliary fields A, k^a and $\varphi_m{}^\alpha$ can be detected. For $T_{\alpha\beta}{}^c$ we refer to ansatz (2.157) and to (2.211). With $\sigma_{ab}{}^\gamma$ and $t_{ab}{}^c$ given by (2.195) and (2.193) the supertorsion components read (cf. also decomposition (2.159) and (2.160))

$$T_{ab}{}^\gamma| = \sigma_{ab}{}^\gamma, \qquad T_{ab}{}^c| = t_{ab}{}^c, \tag{2.225}$$

$$T^+{}_\beta{}^\gamma| = \frac{i}{2}A(\Gamma^3\gamma^3)_\beta{}^\gamma, \qquad T^+{}_\beta{}^c| = 2i\varphi^c{}_\beta, \tag{2.226}$$

the superfield S of ansatz (2.157) for $F_{\alpha\beta}$ to lowest order is

$$S| = (1-k^2)A, \tag{2.227}$$

and for the supercurvature components with f_{ab} given by (2.194) (cf. also (2.158))

$$F_{ab}| = f_{ab}, \tag{2.228}$$

$$F^+{}_\alpha| = 2A\epsilon^{cb}(\Gamma_b\varphi_c)_\alpha - 2iA\varphi^b{}_\alpha k_b + 2i(\Gamma^3\gamma^3\sigma)_\alpha \tag{2.229}$$

is obtained.

The calculation, simplification and analysis of the model is outside the scope of our present work. The complexity of these results suggested the treatment of supergravity along a different path (cf. Chapter 4 below). We note that in (2.193)–(2.196) always the combination $\omega_a - Ak_a$ appears. Actually this combination will be found to be the important one below in the PSM approach (Chapter 4).

Chapter 3

PSM Gravity and its Symplectic Extension

The PSM in general and its relation to two-dimensional gravity is presented. The method to solve the e.o.m.s of general PSMs is derived and the symplectic extension of the two-dimensional gravity PSM is constructed.

3.1 PSM Gravity

A large class of gravity models in two dimensions can be written in first order form

$$L = \int_{\mathcal{M}} (DX^a)e_a + (d\phi)\omega + \frac{1}{2}V\epsilon^{ba}e_a e_b, \tag{3.1}$$

where the potential $V = V(\phi, Y)$ is a function of ϕ and Y with $Y = \frac{1}{2}X^a X_a$, and the covariant exterior derivative is given by

$$DX^a = dX^a + X^b\omega\epsilon_b{}^a. \tag{3.2}$$

Particular choices of V yield various gravity models, among which spherically reduced gravity is an important physical example.

This action is of the form of a Poisson Sigma Model [57, 59, 79, 105, 106] (cf. also [107–111]),

$$L = \int_{\mathcal{M}} dX^i A_i + \frac{1}{2}\mathcal{P}^{ij}A_j A_i. \tag{3.3}$$

The coordinates on the target space $\mathcal{N}$ are denoted by $X^i = (X^a, \phi)$. The same symbols are used to denote the mapping from $\mathcal{M}$ to $\mathcal{N}$, therefore $X^i = X^i(x^m)$. In this sloppy notation the dX^i in the integral above stand for the pull-back of the target space differentials $dX^i = dx^m\partial_m X^i$, and A_i are 1-forms on $\mathcal{M}$ with values in the cotangent space of $\mathcal{N}$. The Poisson

tensor $\mathcal{P}^{ij}$ is an antisymmetric bi-vector field on the target space $\mathcal{N}$, which fulfills the Jacobi identity

$$J^{ijk} = \mathcal{P}^{il}\partial_l \mathcal{P}^{jk} + \text{cycl}(ijk) = 0. \tag{3.4}$$

For the gravity model (3.1) we obtain

$$\mathcal{P}^{ab} = V\epsilon^{ab}, \qquad \mathcal{P}^{a\phi} = X^b \epsilon_b{}^a. \tag{3.5}$$

The equations of motion are

$$dX^i + \mathcal{P}^{ij}A_j = 0, \tag{3.6}$$

$$dA_i + \frac{1}{2}(\partial_i \mathcal{P}^{jk})A_k A_j = 0, \tag{3.7}$$

and the symmetries of the action are found to be

$$\delta X^i = \mathcal{P}^{ij}\epsilon_j, \qquad \delta A_i = -d\epsilon_i - (\partial_i \mathcal{P}^{jk})\epsilon_k A_j. \tag{3.8}$$

Note that $\epsilon'_j = \epsilon_j + \partial_j C$ leads to the same transformations of X^i. Under the local symmetries $\epsilon_i = \epsilon_i(x^m)$ the action transforms into a total derivative, $\delta L = \int d(dX^i \epsilon_i)$. We get Lorentz transformations with parameter $l = l(x^m)$ when making the particular choice $\epsilon_i = (\epsilon_a, \epsilon_\phi) = (0, l)$. We can also represent any infinitesimal diffeomorphism on $\mathcal{M}$, given by $\delta x^m = -\xi^m$, by symmetry transformations with parameter $\epsilon_i = \xi^m A_{mi} = (\xi^m e_{ma}, \xi^m \omega_m)$. These symmetries yield the usual transformation rules for the fields back, $\delta X^i = -\xi^m \partial_m X^i$ and $\delta A_{mi} = -\xi^n \partial_n A_{mi} - (\partial_m \xi^n)A_{ni}$, when going on shell. The transformations with parameter $\epsilon_i = (\epsilon_a, 0)$ reveals the zweibeins e_a to be the corresponding gauge fields, $\delta e_a = -d\epsilon_a + \cdots$, leading us to the notion of 'local translations' for this symmetry.

3.2 Conserved Quantity

If $\mathcal{P}^{ij}$ is not of full rank, then we have functions C so that

$$\{X^i, C\} = \mathcal{P}^{ij}\partial_j C = 0. \tag{3.9}$$

In the case of PSM gravity there is only one such function $C = C(\phi, Y)$ given by its defining equation (3.9)

$$C' - V\dot{C} = 0, \tag{3.10}$$

where $C' = \partial_\phi C$ and $\dot{C} = \partial_Y C$.

From

$$dC = dX^i \partial_i C = -\mathcal{P}^{ij}A_j \partial_i C = 0, \tag{3.11}$$

$V=0$	$C=f\,(Y)$
$V=v_0(\phi)$	$C=f\left(Y+\int_0^\phi v_0(x)dx\right)$
$V=v_0(\phi)+v_1 Y$	$C=f\left(Ye^{v_1\phi}+\int_0^\phi e^{v_1 x}v_0(x)dx\right)$
$V=v_0(\phi)+v_1(\phi)Y$	$C=f\left(Ye^{\int_0^\phi v_1(x')dx'}+\int_0^\phi e^{\int_0^x v_1(x')dx'}v_0(x)dx\right)$

Table 3.1: Conserved Quantity C

where the PSM e.o.m. (3.6) and condition (3.9) were employed, we derive that $dC=0$ on shell. Another view is to take a particular linear combination of (3.6),

$$(dX^i+\mathcal{P}^{ij}A_j)\partial_i C = dX^i\partial_i C = dC, \tag{3.12}$$

which immediately shows that $dC=0$ is an equation of motion. This result can also be derived using the second Noether theorem and making a local transformation with parameter ϵ_C,

$$\delta X^i = 0, \qquad \delta A_i = -(\partial_i C)\, d\epsilon_C, \tag{3.13}$$

for which we get $\delta L = -\int d(dC\epsilon_C)$. This transformation differs from the symmetries (3.8) with parameter $\epsilon_i = (\partial_i C)\epsilon_C$ only by a term proportional to the equations of motion. One can also use the first Noether theorem to establish the same results. Although the symmetry which yields the conserved quantity is already a local one, because $dC=0$ is an e.o.m., one can use a trick and perform a rigid transformation instead. The transformation

$$\delta X^i = 0, \qquad \delta A_i = -(\partial_i C)\, df\epsilon \tag{3.14}$$

with an arbitrary function $f=f(\phi,Y)$ and a rigid transformation parameter ϵ leaves the action (3.3) invariant. Furthermore, the e.o.m.s transform in a total derivative dC, stating that the conserved charge is C [143]. We will give a more geometrical meaning to this symmetry later on. Note that this is a symmetry which acts on the one forms A_i, but has no influence on the coordinates X^i.

3.3 Solving the Equations of Motion

Assuming for simplicity that there is only one function $C(X^i)$, then the equation $C(X^i)=\underline{C}\in\mathbb{R}$ solves one of the differential equations (3.6), but how can we solve the remaining ones? This is easily done by making the coordinate transformation $(X^i)=(X^{++},X^{\mathrm{i}})\to(\underline{C},\underline{X}^{\mathrm{i}})$,

$$\underline{C} = C(X^{++},X^{\mathrm{l}}), \tag{3.15}$$

$$\underline{X}^{\mathrm{i}} = X^{\mathrm{i}}. \tag{3.16}$$

Notice the difference between the index i and $\mathtt{i}$ and that $X^{\mathtt{i}}$ stands for (X^{--},ϕ). For $A_i = (A_{++}, A_{\mathtt{i}})$, using $A_i = \frac{\partial \underline{X}^j}{\partial X^i}\underline{A}_j$, we get

$$A_{++} = (\partial_{++}C)\underline{A}_C, \qquad A_{\mathtt{i}} = \underline{A}_{\mathtt{i}} + (\partial_{\mathtt{i}}C)\underline{A}_C, \tag{3.17}$$

and for the Poisson tensor in the new coordinate system we have

$$\underline{\mathcal{P}}^{C\mathtt{j}} = 0, \qquad \underline{\mathcal{P}}^{\mathtt{ij}} = \mathcal{P}^{\mathtt{ij}}. \tag{3.18}$$

The reduced Poisson tensor of the subspace spanned by $(X^{\mathtt{i}})$ is invertible. The inverse defined by $\underline{\mathcal{P}}^{\mathtt{lj}}\underline{\Omega}_{\mathtt{li}} = \delta_{\mathtt{i}}{}^{\mathtt{j}}$ supplies this subspace with the symplectic two form $\underline{\Omega} = \frac{1}{2}d\underline{X}^{\mathtt{i}}d\underline{X}^{\mathtt{j}}\underline{\Omega}_{\mathtt{ji}}$, which, as a consequence of the Jacobi identity of the Poisson tensor, is closed on shell, $d\underline{\Omega} = d\underline{C}\gamma$ (evaluated on the target space, where it is nontrivial). The action reads in the $(\underline{C}, \underline{X}^{\mathtt{i}})$ coordinate system of the target space

$$L = \int_{\mathcal{M}} d\underline{C}\,\underline{A}_C + d\underline{X}^{\mathtt{i}}\underline{A}_{\mathtt{i}} + \frac{1}{2}\underline{\mathcal{P}}^{\mathtt{ji}}\underline{A}_{\mathtt{i}}\underline{A}_{\mathtt{j}}. \tag{3.19}$$

The e.o.m.s by the variations $\delta\underline{A}_C$ and $\delta\underline{A}_{\mathtt{i}}$,

$$d\underline{C} = 0, \qquad d\underline{X}^{\mathtt{i}} + \underline{\mathcal{P}}^{\mathtt{ij}}\underline{A}_j = 0 \tag{3.20}$$

are then easily solved by $\underline{C} = \text{const}$, as we already knew, and by

$$\underline{A}_{\mathtt{i}} = -d\underline{X}^{\mathtt{j}}\underline{\Omega}_{\mathtt{ji}}. \tag{3.21}$$

In order to investigate the remaining e.o.m.s, following from the variations $\delta\underline{C}$ and $\delta\underline{X}^{\mathtt{i}}$, we first insert the solutions for the $\underline{A}_{\mathtt{i}}$ back into the action, yielding

$$L = \int_{\mathcal{M}} d\underline{C}\,\underline{A}_C - \frac{1}{2}d\underline{X}^{\mathtt{i}}d\underline{X}^{\mathtt{j}}\underline{\Omega}_{\mathtt{ji}}, \tag{3.22}$$

and then derive the equations

$$\delta\underline{A}_C: \qquad d\underline{C} = 0, \tag{3.23}$$

$$\delta\underline{C}: \qquad d\underline{A}_C - \frac{1}{2}d\underline{X}^{\mathtt{i}}d\underline{X}^{\mathtt{j}}\,(\underline{\partial}_C\underline{\Omega}_{\mathtt{ji}}) = 0, \tag{3.24}$$

$$\delta\underline{X}^{\mathtt{i}}: \qquad \underline{\partial}_{\mathtt{i}}\underline{\Omega}_{\mathtt{jk}} + \text{cycl}(\mathtt{ijk}) = 0. \tag{3.25}$$

We see (3.24) is the only differential equation which remains. In order to see how it has to be treated we first stick to PSM gravity and take a closer look at the $(\underline{C}, \underline{X}^{--}, \underline{\phi})$, the $(\underline{X}^{++}, \underline{C}, \underline{\phi})$ and the $(\underline{X}^{++}, \underline{X}^{--}, \underline{C})$ coordinate systems in turn.

In the $(\underline{C}, \underline{X}^{--}, \underline{\phi})$ coordinate system the Poisson tensor of PSM gravity is

$$\underline{\mathcal{P}}^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \underline{X}^{--} \\ 0 & -\underline{X}^{--} & 0 \end{pmatrix}. \tag{3.26}$$

The restriction of it to the subspace $\underline{X}^{\mathfrak{i}} = (\underline{X}^{--}, \underline{\phi})$ is invertible, as long as $\underline{X}^{--} \neq 0$, giving this subspace a symplectic structure:

$$\underline{\mathcal{P}}^{\mathfrak{ij}} = \begin{pmatrix} 0 & \underline{X}^{--} \\ -\underline{X}^{--} & 0 \end{pmatrix}, \qquad \underline{\Omega}_{\mathfrak{ij}} = \begin{pmatrix} 0 & \frac{1}{\underline{X}^{--}} \\ -\frac{1}{\underline{X}^{--}} & 0 \end{pmatrix}. \tag{3.27}$$

The solution of the one forms $\underline{A}_{\mathfrak{i}}$ follow immediately from (3.21), $\underline{A}_{--} = d\underline{\phi}/\underline{X}^{--}$ and $\underline{A}_{\phi} = -d\underline{X}^{--}/\underline{X}^{--}$. The remaining equation (3.24) for $\underline{A}_C$ is particularly simple, due to $\underline{\partial}_C \underline{\Omega}_{\mathfrak{ij}} = 0$ in this special coordinate system, yielding $d\underline{A}_C = 0$. This is solved by $\underline{A}_C = -d\underline{F}$, where $\underline{F} = \underline{F}(x)$. Going back to the original (X^{++}, X^{--}, ϕ) coordinate system using (3.17) we get

$$e_{++} = -X_{++}\dot{C}dF, \tag{3.28}$$

$$e_{--} = \frac{d\phi}{X^{--}} - X_{--}\dot{C}dF, \tag{3.29}$$

$$\omega = -\frac{dX^{--}}{X^{--}} - C'dF. \tag{3.30}$$

The generalization of the PSM to the graded case (Chapter 4) will be the basis of a very direct way to obtain 2d supergravity theories. Also the way to obtain the solution for a particular model (Section 4.7) will essentially follow the procedure described here.

3.4 Symplectic Extension of the PSM

The solution of the PSM e.o.m.s in the $(\underline{X}^{++}, \underline{C}, \underline{\phi})$ coordinate system provides no new insight, it is similar to the one before, but in the $(\underline{X}^{++}, \underline{X}^{--}, \underline{C})$ system things turn out to be more difficult. For the Poisson tensor we obtain

$$\underline{\mathcal{P}}^{ij} = \begin{pmatrix} 0 & \underline{V} & 0 \\ -\underline{V} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.31}$$

The restriction to the subspace $(\underline{X}^{++}, \underline{X}^{--})$ gives

$$\underline{\mathcal{P}}^{\mathfrak{ij}} = \begin{pmatrix} 0 & \underline{V} \\ -\underline{V} & 0 \end{pmatrix}, \qquad \underline{\Omega}_{\mathfrak{ij}} = \begin{pmatrix} 0 & \frac{1}{\underline{V}} \\ -\frac{1}{\underline{V}} & 0 \end{pmatrix}. \tag{3.32}$$

The difficulties stem from the fact that $\underline{\partial}_C \underline{\Omega}_{\mathfrak{ij}} \neq 0$, therefore yielding for (3.24) the equation $d\underline{A}_C - d\underline{X}^{--} d\underline{X}^{++} \underline{\partial}_C (\frac{1}{\underline{V}}) = 0$. Finding a solution for this equation in an elegant way is the intention of a symplectic extension of the PSM where the Poisson tensor is no longer singular.

3.4.1 Simple Case

Let us first treat the more simple case $\underline{\partial}_C\underline{\Omega}_{\mathrm{ij}} = 0$ using the coordinate system $(\underline{C}, \underline{X}^{--}, \underline{\phi})$ as an example. The solution of (3.24) is $\underline{A}_C = -d\underline{F}$, the minus sign is conventional. Inserting this solution back into action (3.22) we get

$$L = \int_{\mathcal{M}} -d\underline{C}d\underline{F} - \frac{1}{2}d\underline{X}^{\mathrm{i}}d\underline{X}^{\mathrm{j}}\underline{\Omega}_{\mathrm{ji}}. \tag{3.33}$$

This directly suggests to extend the target space by adding a new target space coordinate $\underline{F}$. Using $\underline{X}^I = (\underline{F}, \underline{C}, \underline{X}^{--}, \underline{\phi})$ as coordinate system we write the action in the form

$$L = \int_{\mathcal{M}} -\frac{1}{2}d\underline{X}^I d\underline{X}^J \underline{\Omega}_{JI}, \tag{3.34}$$

from which we can immediately read off the extended symplectic matrix and calculate its inverse,

$$\underline{\Omega}_{IJ} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\underline{X}^{--}} \\ 0 & 0 & -\frac{1}{\underline{X}^{--}} & 0 \end{pmatrix}, \quad \underline{\mathcal{P}}^{IJ} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{X}^{--} \\ 0 & 0 & -\underline{X}^{--} & 0 \end{pmatrix}. \tag{3.35}$$

Now let's take a look at the corresponding action in PSM form. When using $\underline{F}$ as target space coordinate it is also necessary to introduce its corresponding one form $\underline{A}_F$, therefore with $\underline{A}_{--}$ and $\underline{A}_\phi$ already eliminated we get the equivalent action

$$L = \int_{\mathcal{M}} d\underline{F}\,\underline{A}_F + d\underline{C}\,\underline{A}_C - \underline{A}_F\underline{A}_C - \frac{1}{2}d\underline{X}^{\mathrm{i}}d\underline{X}^{\mathrm{j}}\underline{\Omega}_{\mathrm{ji}}, \tag{3.36}$$

from which we immediately derive the e.o.m.s

$$\delta\underline{A}_C: \qquad d\underline{C} - \underline{A}_F = 0, \tag{3.37}$$
$$\delta\underline{A}_F: \qquad d\underline{F} + \underline{A}_C = 0, \tag{3.38}$$
$$\delta\underline{C}: \qquad d\underline{A}_C = 0, \tag{3.39}$$
$$\delta\underline{F}: \qquad d\underline{A}_F = 0. \tag{3.40}$$

The general solution in the symplectic case can be immediately written down,

$$\underline{A}_I = -d\underline{X}^J\underline{\Omega}_{JI}. \tag{3.41}$$

We see, gauge-fixing $\underline{A}_F = 0$ yields our original model back. Of course, field equations (3.38) and (3.39) only fit when $\underline{\partial}_C\underline{\Omega}_{\mathrm{ij}} = 0$. If this is not so,

(3.39) would read $d\underline{A}_C - \frac{1}{2}d\underline{X}^{\mathrm{i}}d\underline{X}^{\mathrm{j}}\underline{\partial}_C\underline{\Omega}_{\mathrm{ji}} = 0$ and (3.38) has to be extended by adding some more terms, which means $\underline{\mathcal{P}}^{iF} \neq 0$, or when inverting the Poisson tensor $\underline{\Omega}_{Ci} \neq 0$. Note that the extended Poisson tensor (3.35) already fulfills the Jacobi identity $\underline{\mathcal{P}}^{IL}\underline{\partial}_L\underline{\mathcal{P}}^{JK} + \mathrm{cycl}(IJK) = 0$ and that its inverse, the symplectic matrix, obeys $\underline{\partial}_I\underline{\Omega}_{JK} + \mathrm{cycl}(IJK) = 0$, which is just the statement that the symplectic form on the extended target space $\Omega = \frac{1}{2}d\underline{X}^I d\underline{X}^J\underline{\Omega}_{JI}$ is closed on the target space, $d\Omega = 0$. And it is just the closure of the symplectic form, or, when considering the dual problem, the Jacobi identity, which determines $\underline{\Omega}_{Ci}$ and/or $\underline{\mathcal{P}}^{iF}$ in the case $\underline{\partial}_C\underline{\Omega}_{\mathrm{ij}} \neq 0$, as we demonstrate in a minute.

Next we will investigate the new symmetry transformations we have implicitly added to the extended action. From the transformations $\delta\underline{X}^I = \underline{\mathcal{P}}^{IJ}\underline{\epsilon}_J$ and $\delta\underline{A}_I = -d\underline{\epsilon}_I + \underline{\partial}_I\underline{\mathcal{P}}^{JK}\underline{A}_K\underline{\epsilon}_J$ we first look at the one with parameter $\underline{\epsilon}_C$,

$$\delta\underline{F} = \underline{\epsilon}_C, \qquad \delta\underline{A}_C = -d\underline{\epsilon}_C. \tag{3.42}$$

Although this symmetry was originally available in the gauge potential sector only, $\delta\underline{A}_C = -d\underline{\epsilon}_C$, it can now be interpreted as a transformation in the extended target space, $\delta\underline{F} = \underline{\epsilon}_C$. Transforming back to the coordinate system $(F, X^{++}, X^{--}, \phi)$, which is the original one extended by $F = \underline{F}$, using (3.17) shows that

$$\delta A_i = -(\partial_i C)d\underline{\epsilon}_C, \tag{3.43}$$

is indeed the gauge symmetry of the extended action which is the local version of (3.14) corresponding to the conserved quantity C of the original action. Furthermore, any departure from $\underline{A}_F = 0$ can be made by a symmetry transformation with parameter $\underline{\epsilon}_F$,

$$\delta\underline{C} = -\underline{\epsilon}_F, \qquad \delta\underline{A}_F = -d\underline{\epsilon}_F. \tag{3.44}$$

For completeness we write down the extension in the $(\underline{F}, \underline{X}^{++}, \underline{C}, \underline{\phi})$ coordinate system too:

$$\underline{\Omega}_{IJ} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\underline{X}^{++}} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{\underline{X}^{++}} & 0 & 0 \end{pmatrix}, \quad \underline{\mathcal{P}}^{IJ} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\underline{X}^{++} \\ -1 & 0 & 0 & 0 \\ 0 & \underline{X}^{++} & 0 & 0 \end{pmatrix}. \tag{3.45}$$

3.4.2 Generic Case

As already promised, we are going to solve (3.24) in the $(\underline{F}, \underline{X}^{++}, \underline{X}^{--}, \underline{C})$ coordinate system by extending the target space and looking for an appro-

priate Poisson tensor. The above considerations suggest the ansatz

$$\underline{\mathcal{P}}^{IJ} = \begin{pmatrix} 0 & \underline{\mathcal{P}}^{F++} & \underline{\mathcal{P}}^{F--} & 1 \\ -\underline{\mathcal{P}}^{F++} & 0 & V & 0 \\ -\underline{\mathcal{P}}^{F--} & -V & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.46}$$

or equivalently for the symplectic form

$$\underline{\Omega}_{IJ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{V} & -\underline{\Omega}_{C++} \\ 0 & -\frac{1}{V} & 0 & -\underline{\Omega}_{C--} \\ -1 & \underline{\Omega}_{C++} & \underline{\Omega}_{C--} & 0 \end{pmatrix}. \tag{3.47}$$

The quantities $\underline{\mathcal{P}}^{Fb}$ and $\underline{\Omega}_{Cb}$ should not depend on the coordinate $\underline{F}$. We can further expand $\underline{\Omega}_{Cb}$ in a Lorentz covariant way, $\underline{\Omega}_{Cb} = \frac{1}{2}A\underline{X}_b + \frac{1}{2}B\underline{X}^a\epsilon_{ab}$. The closure relation for Ω by $\underline{J}_{IJK} = \underline{\partial}_{[I}\underline{\Omega}_{JK]}$ yields

$$-\frac{1}{2}\epsilon^{ba}\underline{J}_{Cab} = \underline{Y}(\underline{\partial}_Y B) + B + \underline{\partial}_C(\frac{1}{V}) = 0. \tag{3.48}$$

Then, after inserting $B = \frac{G}{\underline{Y}}$, we arrive at

$$\underline{\partial}_Y G + \underline{\partial}_C(\frac{1}{V}) = 0. \tag{3.49}$$

This equation is solved by

$$G = \frac{1}{C'} + g(\underline{C}), \tag{3.50}$$

as can be seen easily by going back to our original coordinate system, thus using $\underline{\partial}_Y = \partial_Y - \frac{1}{V}\partial_\phi$, $\underline{\partial}_C = \frac{1}{C'}\partial_\phi$ and $V = \frac{C'}{C}$. We set $A = 0$, which can be reached by a coordinate transformation of the type $\underline{F} \to \underline{F} + f(\underline{X}^i, \underline{C})$ as we see later, and calculate $\underline{\mathcal{P}}^{Fb}$ using $\underline{\Omega}_{Cb}$,

$$\underline{\Omega}_{Cb} = \frac{1}{2\underline{Y}}G\underline{X}^a\epsilon_{ab}, \qquad \underline{\mathcal{P}}^{Fb} = \frac{1}{2\underline{Y}}GV\underline{X}^b, \tag{3.51}$$

and then go back to our original coordinate system, in which we have, in addition to (3.5),

$$\mathcal{P}^{F\phi} = 0, \qquad \mathcal{P}^{Fb} = \frac{1}{2Y}GVX^b. \tag{3.52}$$

The homogeneous part in the solution G can also be set to zero, thus we finally have the extended action

$$L^{\text{ext}} = \int_{\mathcal{M}} (dF)A_F + (DX^a)e_a + (d\phi)\omega + \frac{1}{2}V\epsilon^{ba}e_a e_b + \frac{1}{X^a X_a \dot{C}}X^b e_b A_F. \tag{3.53}$$

This is a generalization of 2d gravity with $U(1)$ gauge field A; there V is a function of ϕ, Y and F but the last term is not present [115]. The original model (3.1) is found again when choosing the gauge $A_F = 0$. Of course the gauge $A_F = 0$ restricts the target space to the surface $C(X^i) = \text{const}$, as can be seen immediately from the solution of the e.o.m.s of model (3.53), given by $A_I = -dX^J \Omega_{JI}$, where Ω_{JI} is the inverse of the Poisson tensor,

$$\Omega_{Fi} = \partial_i C, \qquad \Omega_{\phi b} = \frac{1}{2Y} X^a \epsilon_{ab}, \qquad \Omega_{ab} = 0, \tag{3.54}$$

which is given by

$$A_F = dC, \tag{3.55}$$

$$e_a = -dF X_a \dot{C} - d\phi \frac{1}{2Y} X^b \epsilon_{ba}, \tag{3.56}$$

$$\omega = -dF C' + \frac{1}{2Y} dX^a X^b \epsilon_{ba}. \tag{3.57}$$

3.4.3 Uniqueness of the Extension

We now have a look at the various possibilities one has in extending the action. We will see, by using Casimir-Darboux coordinates, that all these possibilities are connected by coordinate transformations of particular types. The simplest case of a symplectic extension is the extension of a model in Casimir-Darboux coordinates. Denoting the coordinates by $X^i = (C, Q, P)$ we have

$$L^{\text{CD}} = \int_{\mathcal{M}} dC A_C + dQ A_Q + dP A_P - A_Q A_P \tag{3.58}$$

and the Poisson tensor reads

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{3.59}$$

The e.o.m.s for A_Q and A_P can be read off immediately. The remaining equations are $dC = 0$ and $dA_C = 0$, where the latter is solved by $A_C = -dF$. We also see that the same equations can be derived from the extended action

$$L^{\text{ext}} = \int_{\mathcal{M}} dF A_F + dC A_C - A_F A_C + dQ A_Q + dP A_P - A_Q A_P \tag{3.60}$$

as long as we restrict the solution to the surface $C = \text{const}$ or alternatively gauge $A_F = 0$.

The extended target space is uniquely determined by the conditions $\partial_F \mathcal{P}^{IJ} = 0$, and the Jacobi identity $J^{IJK} = 0$. One can easily see this

by making the ansatz in the extended target space $X^I = (F, C, Q, P)$

$$\mathcal{P}^{IJ} = \begin{pmatrix} 0 & \mathcal{P}^{FC} & \mathcal{P}^{FQ} & \mathcal{P}^{FP} \\ -\mathcal{P}^{FC} & 0 & 0 & 0 \\ -\mathcal{P}^{FQ} & 0 & 0 & 1 \\ -\mathcal{P}^{FP} & 0 & -1 & 0 \end{pmatrix}. \tag{3.61}$$

From the Jacobi identities $J^{FCQ} = J^{FCP} = 0$ we get $\mathcal{P}^{FC} = \kappa(C)$. Using a coordinate transformation of the form $\underline{C} = g(C)$ we can set $\mathcal{P}^{FC} = 1$. This can be done because any redefinition of C does not change the original model. The remaining Jacobi identity $J^{FQP} = 0$ yields $\partial_Q \mathcal{P}^{FQ} + \partial_P \mathcal{P}^{FP} = 0$, but any solution of this equation can be put to zero by a coordinate transformation $\underline{F} = F + f(C, Q, P)$. The one forms A_C, A_Q and A_F get then contributions of A_F, but on the surface $C = \text{const}$ we have $A_F = 0$, therefore this doesn't change the model too. We conclude that the conditions $\partial_F \mathcal{P}^{IJ} = 0$ and $J^{IJK} = 0$ provide a coordinate independent way to extend the target space and to supply it with a symplectic structure.

3.5 Symplectic Geometry

Symplectic form and Poisson tensor read

$$\Omega = \frac{1}{2} dX^I dX^J \Omega_{JI}, \qquad \mathcal{P} = \frac{1}{2} \mathcal{P}^{IJ} \partial_J \partial_I, \tag{3.62}$$

where $\Omega_{IJ} = -\Omega_{JI}$, $\mathcal{P}^{IJ} = -\mathcal{P}^{JI}$ and $\mathcal{P}^{IJ} \Omega_{IK} = \delta_K{}^J$. The related Poisson bracket of functions becomes

$$\{f, g\} = \mathcal{P}^{IJ} (\partial_J g)(\partial_I f). \tag{3.63}$$

The symplectic form defines an isomorphism between vectors an covectors. Let $v = v^I \partial_I$ be a vector field, then it's correspondent 1-form A_v is

$$A_v = v^\flat = v \rfloor \Omega = dX^I v^J \Omega_{JI}. \tag{3.64}$$

The vector corresponding to a 1-form $\alpha = dX^I \alpha_I$ is

$$\alpha^\sharp = \mathcal{P}^{IJ} \alpha_J \partial_I. \tag{3.65}$$

We have $(v^\flat)^\sharp = v$ and $(\alpha^\sharp)^\flat = \alpha$ of course. Now the Poisson bracket of the 1-forms α and β can be defined by

$$\{\alpha, \beta\} = [\alpha^\sharp, \beta^\sharp]^\flat, \tag{3.66}$$

and the antisymmetric scalar product of vectors reads

$$[u|v] = u^I v^J \Omega_{JI}. \tag{3.67}$$

Hamiltonian vector fields become

$$V_f = (df)^\sharp = \{\,.\,, f\} = \mathcal{P}^{IJ}(\partial_J f)\partial_I. \tag{3.68}$$

(Note: $V_f \rfloor \Omega = df$). The commutator of Hamiltonian vector fields is

$$[V_f, V_g] = V_{\{g,f\}}, \tag{3.69}$$

and the antisymmetric scalar product of Hamiltonian vector fields reads

$$[V_f | V_g] = \{g, f\}. \tag{3.70}$$

On the basis of these formulae the Lie derivative of functions in the direction of Hamiltonian vector fields becomes

$$\mathcal{L}_{V_f}(g) = V_f(g) = \{g, f\} = -V_g(f) = -\mathcal{L}_{V_g}(f), \tag{3.71}$$

and the Lie derivative of vectors in the direction of Hamiltonian vector fields

$$\mathcal{L}_{V_f}(u) = [V_f, u]. \tag{3.72}$$

Symplectic form and Poisson tensor obey

$$\mathcal{L}_{V_f}(\Omega) = \mathcal{L}_{V_f}(\mathcal{P}) = 0. \tag{3.73}$$

3.6 Symplectic Gravity

In terms of the coordinates $X^I = (F, X^a, \phi)$ the new components of the Poisson tensor are $\mathcal{P}^{Fb} = \frac{1}{2Y\dot{C}}X^b$ and $\mathcal{P}^{F\phi} = 0$. Its inverse, the symplectic form, consists of the components $\Omega_{Fi} = (\partial_i C)$ and $\Omega_{\phi b} = \frac{1}{2Y}X^c\epsilon_{cb}$.

$$\mathcal{P}^{IJ} = \left(\begin{array}{c|cc} 0 & \mathcal{P}^{Fb} & \mathcal{P}^{F\phi} \\ \hline \mathcal{P}^{aF} & \mathcal{P}^{ab} & \mathcal{P}^{a\phi} \\ \mathcal{P}^{\phi F} & \mathcal{P}^{\phi b} & 0 \end{array}\right) = \left(\begin{array}{c|cc} 0 & \frac{1}{2Y\dot{C}}X^b & 0 \\ \hline -\frac{1}{2Y\dot{C}}X^a & V\epsilon^{ab} & X^c\epsilon_c{}^a \\ 0 & -X^c\epsilon_c{}^b & 0 \end{array}\right), \tag{3.74}$$

$$\Omega_{IJ} = \left(\begin{array}{c|cc} 0 & \Omega_{Fb} & \Omega_{F\phi} \\ \hline \Omega_{aF} & \Omega_{ab} & \Omega_{a\phi} \\ \Omega_{\phi F} & \Omega_{\phi b} & 0 \end{array}\right) = \left(\begin{array}{c|cc} 0 & \partial_b C & \partial_\phi C \\ \hline -\partial_a C & 0 & -\frac{1}{2Y}X^c\epsilon_{ca} \\ -\partial_\phi C & \frac{1}{2Y}X^c\epsilon_{cb} & 0 \end{array}\right). \tag{3.75}$$

$$\mathcal{P}^{IJ} = \left(\begin{array}{c|ccc} 0 & \frac{1}{2X^{--}\dot{C}} & \frac{1}{2X^{++}\dot{C}} & 0 \\ \hline -\frac{1}{2X^{--}\dot{C}} & 0 & V & -X^{++} \\ -\frac{1}{2X^{++}\dot{C}} & -V & 0 & X^{--} \\ 0 & X^{++} & -X^{--} & 0 \end{array}\right), \tag{3.76}$$

$$\Omega_{IJ} = \left(\begin{array}{c|ccc} 0 & X^{--}\dot{C} & X^{++}\dot{C} & C' \\ \hline -X^{--}\dot{C} & 0 & 0 & -\frac{1}{2X^{++}} \\ -X^{++}\dot{C} & 0 & 0 & \frac{1}{2X^{--}} \\ -C' & \frac{1}{2X^{++}} & -\frac{1}{2X^{--}} & 0 \end{array}\right). \tag{3.77}$$

$$\det(\Omega_{IJ}) = \dot{C}^2, \qquad \det(\mathcal{P}^{IJ}) = \frac{1}{\det(\Omega_{IJ})} = \frac{1}{\dot{C}^2}. \tag{3.78}$$

Let Φ: $\mathcal{M} \to \mathcal{N}$ and (x^m) be a coordinate system on $\mathcal{M}$. We have the relations for the 1-forms A_I:

$$A_I = A_{\partial_I} = (\partial_I)^\flat = \partial_I \rfloor \Omega = -dX^J \Omega_{JI} \quad \in \Lambda^1(\mathcal{N}), \tag{3.79}$$

$$\Phi^* A_I = -dx^m (\partial_m \Phi^J) \Omega_{JI} = dx^m A_{mI} \quad \in \Lambda^1(\mathcal{M}), \tag{3.80}$$

$$A_{\Phi_* \partial_m} = dX^I (\partial_m \Phi^J) \Omega_{JI} = -dX^I A_{mI} \quad \in \Lambda^1(\mathcal{N}), \tag{3.81}$$

$$A_{mI} = -\partial_I \rfloor (A_{\Phi_* \partial_m}) = -(\partial_m \Phi^J) \Omega_{JI}. \tag{3.82}$$

The PSM Lagrangian can be written as

$$\mathcal{L} = \Phi^* \left(dX^I \wedge (\partial_I \rfloor \Omega) + \frac{1}{2} \mathcal{P}^{IJ} (\partial_J \rfloor \Omega) \wedge (\partial_I \rfloor \Omega) \right), \tag{3.83}$$

and the Hamiltonian flux of C is

$$V_C = (dC)^\sharp = \partial_F. \tag{3.84}$$

Let Φ: $\mathbb{R} \to \mathcal{N}$ be the integral curve $\Phi(\tau)$ of the Hamiltonian vector field V_C. It satisfies the condition $\Phi_* \partial_\tau = V_C$. When using the coordinate representation the Hamiltonian equations

$$\frac{dX^I}{d\tau} = \mathcal{P}^{IJ} \frac{\partial C}{\partial X^J} \tag{3.85}$$

follow, thus $\frac{dF}{d\tau} = 1$ and $\frac{dX^i}{d\tau} = 0$. This suggests to use $(x^m) = (F, x^1)$ as coordinate system on $\mathcal{M}$, because the X^i are then functions of x^1 only, $X^i = X^i(x^1)$. The induced symplectic form on the world sheet $\mathcal{M}$ given by $\omega = \Phi^* \Omega$ in that coordinates reads $\omega = -dF dx^1 (\partial_1 X^i)(\partial_i C) = -dF dx^1 \frac{dC}{dx^1}$, which is zero on the surface $C = \text{const}$.

The surface $C = \text{const}$ in parameter representation using (F, x^1) as coordinate system on $\mathcal{M}$ follows from $\Phi^*(dC) = dx^1((\partial_1 Y)\dot{C} + (\partial_1 \phi) C') = 0$, thus yielding the differential equation

$$\frac{\partial_1 Y}{\partial_1 \phi} = -V(\phi, Y). \tag{3.86}$$

This immediately leads to the $(x^m) = (F, \phi)$ system on $\mathcal{M}$, where we have to solve

$$\frac{dY}{d\phi} = -V(\phi, Y(\phi)). \tag{3.87}$$

3.6.1 Symmetry Adapted Coordinate Systems

The coordinates $(X^a) = (X^{++}, X^{--})$ transform under Lorentz transformations with infinitesimal parameter l according to $\delta X^a = l X^b \epsilon_b{}^a$. It is natural to replace (X^{++}, X^{--}) by new coordinates, one of these being the invariant $Y = \frac{1}{2} X^a X_a = X^{++} X^{--}$. In order to determine the second new coordinate we calculate the finite Lorentz transformations by solving $\frac{dX^a(\lambda)}{d\lambda}|_{\lambda=0} = X^b_{(0)} \epsilon_b{}^a$, for which we get

$$\begin{cases} X^{++}(\lambda) = e^{-\lambda}\, X^{++}_{(0)} \\ X^{--}(\lambda) = e^{\lambda}\, X^{--}_{(0)} \end{cases} . \tag{3.88}$$

Now we see that $\lambda = -\frac{1}{2}\ln(\frac{X^{++}}{X^{--}})$, or $\lambda = -\frac{1}{2}\ln(-\frac{X^{++}}{X^{--}})$ if $\frac{X^{++}}{X^{--}}$ is negative, is an appropriate choice for the remaining new coordinate. This coordinate is best suited, because any Lorentz transformation is a translation in the λ-space.

Therefore, we change from the original coordinates (X^{++}, X^{--}) to symmetry adapted coordinates (Y, λ), where Y is an invariant of the symmetry transformation and λ is in a sense isomorphic to the transformation group, by incorporating the coordinate transformations:

$$X^{++}X^{--} > 0: \qquad \begin{cases} Y = X^{++}X^{--} \\ \lambda = -\dfrac{1}{2}\ln\left(\dfrac{X^{++}}{X^{--}}\right) \end{cases} \tag{3.89}$$

$$X^{++}X^{--} < 0: \qquad \begin{cases} Y = X^{++}X^{--} \\ \lambda = -\dfrac{1}{2}\ln\left(-\dfrac{X^{++}}{X^{--}}\right) \end{cases} \tag{3.90}$$

$$X^{++} = 0,\ X^{--} \neq 0: \qquad \begin{cases} Y = 0 \\ \lambda = \ln\left(X^{--}\right) \end{cases} \tag{3.91}$$

$$X^{++} \neq 0,\ X^{--} = 0: \qquad \begin{cases} Y = 0 \\ \lambda = -\ln\left(X^{++}\right) \end{cases} \tag{3.92}$$

The point $X^{++} = 0$, $X^{--} = 0$ is non-sensitive to Lorentz transformations and therefore cannot be represented in this way.

A short calculation yields the transformed components of the Poisson tensor (3.76) and the symplectic form (3.77). Going from the original coordinates $X^I = (F, X^{++}, X^{--}, \phi)$ to the Lorentz symmetry adapted system $\underline{X}^I = (F, Y, \lambda, \phi)$ with $Y = X^{++}X^{--}$ and $\lambda = -\frac{1}{2}\ln\frac{X^{++}}{X^{--}}$ using

$\underline{\mathcal{P}}^{IJ} = \mathcal{P}^{KL}(\partial_L \underline{X}^J)(\partial_K \underline{X}^I)$ we get

$$\underline{\mathcal{P}}^{IJ} = \begin{pmatrix} 0 & \frac{1}{\dot{C}} & 0 & 0 \\ -\frac{1}{\dot{C}} & 0 & V & 0 \\ 0 & -V & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.93}$$

$$\underline{\Omega}_{IJ} = \begin{pmatrix} 0 & \dot{C} & 0 & C' \\ -\dot{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -C' & 0 & -1 & 0 \end{pmatrix}. \tag{3.94}$$

$$\det(\underline{\Omega}_{IJ}) = \dot{C}^2, \qquad \det(\underline{\mathcal{P}}^{IJ}) = \frac{1}{\dot{C}^2}. \tag{3.95}$$

In flat space where $V = 0$ we have $C = Y$ and the Poisson tensor (3.93) is already in Darboux form. In this case Y is not only Lorentz invariant, but also invariant under spacetime translations. This suggests to use an invariant for both spacetime as well as Lorentz transformations as new coordinate instead of Y, but this is the quantity C. Using (F, C, λ, ϕ) as coordinates of the target space, we arrive at the Darboux form in the curved case too, where the conjugate pairs are

$$\{F, C\} = 1, \qquad \{\lambda, \phi\} = 1, \tag{3.96}$$

and the symplectic form reads

$$\Omega = dCdF + d\phi d\lambda. \tag{3.97}$$

The closure of the symplectic form is now evident, and we can immediately write down the Cartan 1-form θ, which is the potential for the symplectic form, $\Omega = d\theta$, determined up to an exact form:

$$\theta = -CdF + \lambda d\phi \tag{3.98}$$

Chapter 4

Supergravity from Poisson Superalgebras

The method presented in this chapter is able to provide the geometric actions for most general $N = 1$ supergravity in two spacetime dimensions. Our construction implies the possibility of an extension to arbitrary N. This provides a supersymmetrization of any generalized dilaton gravity theory or of any theory with an action being an (essentially) arbitrary function of curvature and torsion.

Technically we proceed as follows: The bosonic part of any of these theories may be characterized by a generically nonlinear Poisson bracket on a three-dimensional target space. In analogy to the given ordinary Lie algebra, we derive all possible $N = 1$ extensions of any of the given Poisson (or W-) algebras. Using the concept of graded Poisson Sigma Models, any extension of the algebra yields a possible supergravity extension of the original theory, local Lorentz and super-diffeomorphism invariance follow by construction. Our procedure automatically restricts the fermionic extension to the minimal one; thus local supersymmetry is realized on-shell. By avoiding a superfield approach of Chapter 2 we are also able to circumvent in this way the introduction of constraints and their solution. Instead, we solve the Jacobi identities of the graded Poisson algebra. The rank associated with the fermionic extension determines the number of arbitrary parameter functions in the solution. In this way for many well-known dilaton theories different supergravity extensions are derived. It turns out that these extensions also may yield restrictions on the range of the bosonic variables.

4.1 Graded Poisson Sigma Model

4.1.1 Outline of the Approach

The PSM formulation of gravity theories allows a direct generalization, yielding possible supergravity theories. Indeed, from this perspective it is suggestive to replace the Minkowski space with its linear coordinates X^a by its superspace analogue, spanned by X^a and (real, i.e. Majorana) spinorial and (one or more) Grassmann-valued coordinates $\chi^{\mathfrak{i}\alpha}$ (where $\mathfrak{i} = 1, \ldots, N$). In the purely bosonic case we required that ϕ generates Lorentz transformations on Minkowski space. We now extend this so that ϕ is the generator of Lorentz transformations on superspace. This implies in particular that besides (1.12) now also

$$\{\chi^{\mathfrak{i}\alpha}, \phi\} = -\frac{1}{2}\chi^{\mathfrak{i}\beta}{\gamma^3}_\beta{}^\alpha, \tag{4.1}$$

has to hold, where $-\frac{1}{2}{\gamma^3}_\beta{}^\alpha$ is the generator of Lorentz transformations in the spinorial representation. For the choice of the γ-matrices and further details on notation and suitable identities we refer to Appendix B.

Within the present work we first focus merely on a consistent extension of the original bosonic Poisson algebra to the total superspace. This superspace can be built upon N pairs of coordinates obeying (4.1). Given such a graded Poisson algebra, the corresponding Sigma Model provides a possible N-supergravity extension of the original gravity model corresponding to the purely bosonic sigma model. We shall mainly focus on the construction of a graded Poisson tensor $\mathcal{P}^{IJ}$ for the simplest supersymmetric extension $N = 1$, i.e. on a ('warped') product of the above superspace and the linear space spanned by the generator ϕ. Upon restriction to the bosonic submanifold $\chi^\alpha = 0$, the bracket will be required to coincide with the bracket (1.12) and (1.13) corresponding to the bosonic theory (1.11). Just as the framework of PSMs turns out to provide a fully satisfactory and consistent 2d gravity theory with all the essential symmetries for any given (Lorentz invariant) Poisson bracket (1.12) and (1.13), the framework of graded Poisson Sigma Models (gPSM) will provide possible generalizations for any of the brackets $\mathcal{P}^{ij}$ with a local 'supersymmetry' of the generic type (1.15). In particular, by construction of the general theory (cf [132] or Section 4.2 below) and upon an identification which is a straightforward extension of (1.10), the resulting gravity theory will be invariant automatically with respect to local Lorentz transformations, spacetime diffeomorphisms *and* local supersymmetry transformations. In particular, the Rarita-Schwinger field ψ_α (or $\psi_{\mathfrak{i}\alpha}$, $\mathfrak{i} = 1, \ldots, N$ in the more general case) is seen to enter naturally as the fermionic component of the one-form valued multiplet A_I. Likewise, specializing the local symmetries (1.14) (or rather their generalization to the graded case provided in (4.15) below) to the spinorial part ϵ_α, local supersymmetry transformations of the form (1.15) are found, which, by construction,

are symmetries of the action (In fact, it is here where the graded Jacobi identity for $\mathcal{P}$ enters as an essential ingredient!). Finally, by construction, the bosonic part of the action of the gPSM corresponding to the bracket $\mathcal{P}^{IJ}$ will coincide with (1.11). Thus, for any such a bracket $\mathcal{P}^{IJ}$, the resulting model should allow the interpretation as a permissible supersymmetric generalization of the original bosonic starting point.

The relations (1.12), (4.1) fix the ϕ components of the sought for (graded) Poisson tensor $\mathcal{P}^{IJ}$. We are thus left with determining the remaining components $\mathcal{P}^{AB}$, A and B being indices in the four-dimensional superspace with $X^A = (X^a, \chi^\alpha)$. As will be recapitulated in Section 4.1.2, besides the graded symmetry of the tensor $\mathcal{P}^{IJ}$, the only other requirement it has to fulfill by definition is the graded Jacobi identity. This requires the vanishing of a 3-tensor J^{IJK} (cf. (4.4) below), which may be expressed also as the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}$ of $(\mathcal{P}^{IJ})$ with itself. In this formulation $(\mathcal{P}^{IJ})$ is meant to be the Poisson tensor itself and not its components (abstract indices). It is straightforward to verify (cf. also [132]) that the relations $J^{IJK} = 0$ with at least one of the indices coinciding with the one corresponding to ϕ are satisfied, *iff* $(\mathcal{P}^{AB})$ is a Lorentz covariant 2-tensor,

$$\mathcal{L}_{(\mathcal{P}^{A\phi})}(\mathcal{P}^{AB}) = 0, \tag{4.2}$$

i.e. depending on X^a, χ^α and also on the Lorentz invariant quantity ϕ in a covariant way as determined by its indices. Thus one is left with finding the general solution of $J^{ABC} = 0$ starting from a Lorentz covariant ansatz for $(\mathcal{P}^{AB})$.

Let us note on this occasion that the above considerations do *not* imply that $(\mathcal{P}^{AB})$ forms a bracket on the Super-Minkowski space, a subspace of the target space under consideration. The reason is that the equations $J^{ABC} = 0$ contain also derivatives of $\mathcal{P}^{AB}$ with respect to ϕ: in terms of the Schouten-Nijenhuis bracket, the remaining equations become

$$[(\mathcal{P}^{AB}), (\mathcal{P}^{AB})]_{SN} = (\mathcal{P}^{A\phi}) \wedge (\partial_\phi \mathcal{P}^{AB}), \tag{4.3}$$

where the components of the supervector $(\mathcal{P}^{A\phi})$ are given implicitly by eqs. (1.12) and (4.1) above. So $(\mathcal{P}^{AB})$ defines a graded Poisson bracket for the X^A only if it is independent of ϕ. However, in the present context ϕ-independent Poisson tensors are uninteresting in view of our discussion of actions of the form (1.2).

It should be remarked that given a particular bosonic model and its corresponding bracket, there is by no means a unique graded extension, even for fixed N. Clearly, any (super-)diffeomorphism leaving invariant the bosonic sector as well as the brackets (4.1) applied to a solution of the (graded) Jacobi identities yields another solution. This induces an ambiguity for the choice of a superextension of a given gravity model (1.3) or also (1.1). This is in contrast to the direct application of, say, the superfield formalism

of Howe [46], which when applied to the (necessarily torsionfree) theory (1.3) [125], yields one particular superextension. This now turns out as just one possible choice within an infinite dimensional space of admissible extensions. From one or the other perspective, however, different extensions (for a given N) may be regarded also as effectively equivalent. We shall come back to these issues below.

A final observation concerns the relation of our supersymmetric extensions to 'ordinary' supergravity. From the point of view of the seminal work on the 2d analogue [34, 144] of 4d supergravity our supergravity algebra is 'deformed' by the presence of a dilaton field. Such a feature is known also from the dimensional reduction of supergravity theories in higher dimensions, where one or more dilaton fields arise from the compactification.

4.1.2 Details of the gPSM

In this Section we recollect for completeness some general and elementary facts about graded Poisson brackets and the corresponding Sigma Models. This Section (cf. also Section 2.1.1 and Appendix A, B) also sets the conventions about signs etc. used within the present work, which are adapted to those of [141] and which differ on various instances from those used in [132].

For the construction of the gPSM we take a 2-dimensional base manifold $\mathcal{M}$, also called world sheet or spacetime manifold, with purely bosonic (commutative) coordinates x^m, and the target space $\mathcal{N}$ with coordinates $X^I = (\phi, X^A) = (\phi, X^a, \chi^\alpha)$, ϕ and X^a being bosonic and χ^α fermionic (anticommutative), promoting $\mathcal{N}$ to a supermanifold. The restriction to one Majorana spinor means that only the case $N = 1$ is implied in what follows. To the coordinate functions X^I correspond gauge fields A_I which we identify with the usual Lorentz-connection 1-form ω and the vielbein 1-form e_a of the Einstein-Cartan formalism of gravity and with the Rarita-Schwinger 1-form ψ_α of supergravity according to $A_I = (\omega, e_A) = (\omega, e_a, \psi_\alpha)$. They can be viewed as 1-forms on the base manifold $\mathcal{M}$ with values in the cotangential space of $\mathcal{N}$ and may be collected in the total 1-1-form $A = dX^I A_I = dX^I dx^m A_{mI}$.

As the main structure of the model we choose a Poisson tensor $\mathcal{P}^{IJ} = \mathcal{P}^{IJ}(X)$ on $\mathcal{N}$, which encodes the desired symmetries and the dynamics of the theory to be constructed. Due to the grading of the coordinates of $\mathcal{N}$ it is graded antisymmetric $\mathcal{P}^{IJ} = -(-1)^{IJ}\mathcal{P}^{JI}$ and is assumed to fulfill the graded Jacobi identity ($\vec{\partial}_I = \vec{\partial}/\partial X^I$ is the right derivative) of which we

list also a convenient alternative version

$$
\begin{aligned}
J^{IJK} &= \mathcal{P}^{IL}\vec{\partial}_L\mathcal{P}^{JK} + \mathrm{gcycl}(IJK) && (4.4)\\
&= \mathcal{P}^{IL}\vec{\partial}_L\mathcal{P}^{JK} + \mathcal{P}^{JL}\vec{\partial}_L\mathcal{P}^{KI}(-1)^{I(J+K)} + \mathcal{P}^{KL}\vec{\partial}_L\mathcal{P}^{IJ}(-1)^{K(I+J)} && (4.5)\\
&= 3\mathcal{P}^{I]L}\vec{\partial}_L\mathcal{P}^{[JK} = 0. && (4.6)
\end{aligned}
$$

The relation between the right partial derivative $\vec{\partial}_I$ and the left partial derivative $\overleftarrow{\partial}_I$ for the graded case formally is the same as in (2.3) ($M \to I$). The Poisson tensor defines the Poisson bracket of functions f, g on $\mathcal{N}$,

$$\{f,g\} = (f\overleftarrow{\partial}_J)\mathcal{P}^{JI}(\vec{\partial}_I g), \tag{4.7}$$

implying for the coordinate functions $\{X^I, X^J\} = \mathcal{P}^{IJ}$. With (2.3) the Poisson bracket (4.7) may be written also as

$$\{f,g\} = \mathcal{P}^{JI}(\vec{\partial}_I g)(\vec{\partial}_J f)(-1)^{g(f+J)}. \tag{4.8}$$

This bracket is graded anticommutative,

$$\{f,g\} = -(-1)^{fg}\{g,f\}, \tag{4.9}$$

and fulfills the graded Jacobi identity

$$
\begin{aligned}
\{X^I,\{X^J,X^K\}\}(-1)^{IK} + \{X^J,\{X^K,X^I\}\}(-1)^{JI}&\\
+ \{X^K,\{X^I,X^J\}\}(-1)^{KJ} = 0,& \qquad (4.10)
\end{aligned}
$$

which is equivalent to the graded derivation property

$$\{X^I,\{X^J,X^K\}\} = \{\{X^I,X^J\},X^K\} + (-1)^{IJ}\{X^J,\{X^I,X^K\}\}. \tag{4.11}$$

The PSM action (1.11) generalizes to

$$L^{\mathrm{gPSM}} = \int_{\mathcal{M}} dX^I A_I + \frac{1}{2}\mathcal{P}^{IJ}A_J A_I, \tag{4.12}$$

where in the graded case the sequence of the indices is important. The functions $X^I(x)$ represent a map from the base manifold to the target space in the chosen coordinate systems of $\mathcal{M}$ and $\mathcal{N}$, and dX^I is the shorthand notation for the derivatives $d^{\mathcal{M}}X^I(x) = dx^m\partial_m X^I(x)$ of these functions. The reader may notice the overloading of the symbols X^I which sometimes are used to denote the map from the base manifold to the target space and sometimes, as in the paragraph above, stand for target space coordinates. This carries over to other expressions like dX^I which denote the coordinate differentials $d^{\mathcal{N}}X^I$ on $\mathcal{N}$ and, on other occasions, as in the action (4.12), the derivative of the map from $\mathcal{M}$ to $\mathcal{N}$.

The variation of A_I and X^I in (4.12) yields the gPSM field equations

$$dX^I + \mathcal{P}^{IJ} A_J = 0, \tag{4.13}$$

$$dA_I + \frac{1}{2}(\vec{\partial}_I \mathcal{P}^{JK}) A_K A_J = 0. \tag{4.14}$$

These are first order differential equations of the fields $X^I(x)$ and $A_{mI}(x)$ and the Jacobi identity (4.4) of the Poisson tensor ensures the closure of (4.13) and (4.14). As a consequence of (4.4) the action exhibits the symmetries

$$\delta X^I = \mathcal{P}^{IJ}\epsilon_J, \qquad \delta A_I = -d\epsilon_I - (\vec{\partial}_I \mathcal{P}^{JK})\epsilon_K A_J, \tag{4.15}$$

where corresponding to each gauge field A_I we have a symmetry parameter $\epsilon_I(x)$ with the same grading which is a function of x only. In general, when calculating the commutator of these symmetries, parameters depending on both x and X are obtained. For two parameters $\epsilon_{1I}(x, X)$ and $\epsilon_{2I}(x, X)$

$$(\delta_1\delta_2 - \delta_2\delta_1)\, X^I = \delta_3 X^I, \tag{4.16}$$

$$(\delta_1\delta_2 - \delta_2\delta_1)\, A_I = \delta_3 A_I + \left(dX^J + \mathcal{P}^{JK} A_K\right)(\vec{\partial}_J \vec{\partial}_I \mathcal{P}^{RS})\epsilon_{1S}\epsilon_{2R} \tag{4.17}$$

follows, where $\epsilon_{3I}(x, X)$ of the resulting variation δ_3 are given by the Poisson bracket (or Koszul-Lie bracket) of the 1-forms $\epsilon_1 = dX^I \epsilon_{1I}$ and $\epsilon_2 = dX^I \epsilon_{2I}$, defined according to

$$\epsilon_{3I} = \{\epsilon_2, \epsilon_1\}_I := (\vec{\partial}_I \mathcal{P}^{JK})\epsilon_{1K}\epsilon_{2J} + \mathcal{P}^{JK}\left(\epsilon_{1K}\vec{\partial}_J \epsilon_{2I} - \epsilon_{2K}\vec{\partial}_J \epsilon_{1I}\right). \tag{4.18}$$

Note, that the commutator of the PSM symmetries closes if the Poisson tensor is linear, for non-linear Poisson tensors the algebra closes only on-shell (4.17).

Right and left Hamiltonian vector fields are defined by $\vec{T}^I = \{X^I, \cdot\}$ and $\overleftarrow{T}^I = \{\cdot, X^I\}$, respectively, i.e. by

$$\vec{T}^I \cdot f = \{X^I, f\} = \mathcal{P}^{IJ}(\vec{\partial}_J f), \qquad f \cdot \overleftarrow{T}^I = \{f, X^I\} = (f\overleftarrow{\partial}_J)\mathcal{P}^{JI}. \tag{4.19}$$

The vector fields $\overleftarrow{T}^I$ are the generators of the symmetries, $\delta X^I = X^I \cdot \overleftarrow{T}^J \epsilon_J$. From their commutator the algebra

$$[\overleftarrow{T}^I, \overleftarrow{T}^J] = \overleftarrow{T}^K f_K{}^{IJ}(X) \tag{4.20}$$

follows with the structure functions $f_K{}^{IJ} = (\vec{\partial}_K \mathcal{P}^{IJ})$. Structure constants and therefore Lie algebras are obtained when the Poisson tensor depends only linearly on the coordinates, which is true for Yang-Mills gauge theory and simple gravity models like (anti-)de Sitter gravity.

As in the purely bosonic case the kernel of the graded Poisson algebra determines the so-called Casimir functions C obeying $\{C, X^I\} = 0$. When the co-rank of the bosonic theory—with one Casimir function—is not changed we shall call this case non-degenerate. Then $\mathcal{P}^{\alpha\beta}|$, the bosonic part of the fermionic extension, must be of full rank. For $N = 1$ supergravity and thus one target space Majorana spinor χ^α, the expansion of C in χ^α reads ($\chi^2 = \chi^\alpha\chi_\alpha$, cf. Appendix B)

$$C = c + \frac{1}{2}\chi^2 c_2, \tag{4.21}$$

where c and c_2 are functions of ϕ and $Y \equiv \frac{1}{2}X^a X_a$ only. This assures that the Poisson bracket $\{\phi, C\}$ is zero. From the bracket $\{X^a, C\} = 0$, to zeroth order in χ^α, the defining equation of the Casimir function for pure bosonic gravity PSMs becomes

$$\nabla c := (\partial_\phi - v\partial_Y)\, c = 0. \tag{4.22}$$

This is the well-known partial differential equation of that quantity [59, 79]. The solution of (4.22) for bosonic potentials relevant for kinetic dilaton theories (1.4) can be given by ordinary integration,

$$c(\phi, Y) = Y e^{Q(\phi)} + W(\phi), \tag{4.23}$$

$$Q(\phi) = \int_{\phi_1}^{\phi} Z(\varphi) d\varphi, \qquad W(\phi) = \int_{\phi_0}^{\phi} e^{Q(\varphi)} V(\varphi) d\varphi. \tag{4.24}$$

The new component c_2 is derived by considering the terms proportional to χ^β in the bracket $\{\chi^\alpha, C\} = 0$. Thus c_2 will depend on the specific fermionic extension. In the degenerate case, when $\mathcal{P}^{\alpha\beta}$ is not of full rank, there will be more than one Casimir function, including purely Grassmann valued ones (see Section 4.2.2 and 4.2.2).

4.2 Solution of the Jacobi Identities

As mentioned above, in order to obtain the general solution of the graded Jacobi identities a suitable starting point is the use of Lorentz symmetry in a most general ansatz for $\mathcal{P}^{IJ}$. Alternatively, one could use a simple $\mathcal{P}^{IJ}_{(0)}$ which trivially fulfills (4.4). Then the most general $\mathcal{P}^{IJ}$ may be obtained by a general diffeomorphism in target space. The first route will be followed within this section. We will comment upon the second one in Section 4.3.

4.2.1 Lorentz-Covariant Ansatz for the Poisson-Tensor

Lorentz symmetry determines the mixed components $\mathcal{P}^{A\phi}$ of $\mathcal{P}^{IJ}$,

$$\mathcal{P}^{a\phi} = X^b \epsilon_b{}^a, \qquad \mathcal{P}^{\alpha\phi} = -\frac{1}{2}\chi^\beta \gamma^3{}_\beta{}^\alpha. \tag{4.25}$$

All other components of the Poisson tensor must be Lorentz-covariant (cf. the discussion around (4.2)). Expanding them in terms of invariant tensors η^{ab}, ϵ^{ab}, $\epsilon^{\alpha\beta}$ and γ-matrices yields

$$\mathcal{P}^{ab} = V\epsilon^{ab}, \tag{4.26}$$

$$\mathcal{P}^{\alpha b} = \chi^{\beta} F^{b}{}_{\beta}{}^{\alpha}, \tag{4.27}$$

$$\mathcal{P}^{\alpha\beta} = U\gamma^{3\alpha\beta} + i\widetilde{U}X^c\gamma_c{}^{\alpha\beta} + i\widehat{U}X^c\epsilon_c{}^d\gamma_d{}^{\alpha\beta}. \tag{4.28}$$

The quantities V, U, $\widetilde{U}$ and $\widehat{U}$ are functions of ϕ, Y and χ^2. Due to the anticommutativity of χ^α the dependence on χ^2 is at most linear. Therefore

$$V = v(\phi, Y) + \frac{1}{2}\chi^2\, v_2(\phi, Y) \tag{4.29}$$

depends on two Lorentz-invariant functions v and v_2 of ϕ and Y. An analogous notation will be implied for $U, \widetilde{U}$ and $\widehat{U}$, using the respective lower case letter for the χ-independent component of the superfield and an additional index 2 for the respective χ^2-component. The component (4.27) contains the spinor matrix $F^a{}_\beta{}^\gamma$, which may be first expanded in terms of the linearly independent γ-matrices,

$$F^a{}_\beta{}^\gamma = f^a_{(1)}\delta_\beta{}^\gamma + if^{ab}\gamma_{b\beta}{}^\gamma + f^a_{(3)}\gamma^3{}_\beta{}^\gamma. \tag{4.30}$$

The Lorentz-covariant coefficient functions in (4.30) are further decomposed according to

$$f^a_{(1)} = f_{(11)}X^a - f_{(12)}X^b\epsilon_b{}^a, \tag{4.31}$$

$$f^a_{(3)} = f_{(31)}X^a - f_{(32)}X^b\epsilon_b{}^a, \tag{4.32}$$

$$f^{ab} = f_{(s)}\eta^{ab} + f_{(t)}X^aX^b - f_{(h)}X^c\epsilon_c{}^aX^b + f_{(a)}\epsilon^{ab}. \tag{4.33}$$

The eight Lorentz-invariant coefficients $f_{(11)}$, $f_{(12)}$, $f_{(31)}$, $f_{(32)}$, $f_{(s)}$, $f_{(t)}$, $f_{(h)}$ and $f_{(a)}$ are functions of ϕ and Y only. The linearity in χ^α of (4.27) precludes any χ^2 term in (4.30).

Below it will turn out to be convenient to use a combined notation for the bosonic and the χ^2-dependent part of $\mathcal{P}^{\alpha\beta}$,

$$\mathcal{P}^{\alpha\beta} = v^{\alpha\beta} + \frac{1}{2}\chi^2 v_2^{\alpha\beta}, \tag{4.34}$$

where $v^{\alpha\beta}$ and $v_2^{\alpha\beta}$ are particular matrix-valued functions of ϕ and X^a, namely, in the notation above (cf. also Appendix B.1 for the definition of X^{++} and X^{--}),

$$v^{\alpha\beta} = \begin{pmatrix} \sqrt{2}X^{++}(\tilde{u} - \hat{u}) & -u \\ -u & \sqrt{2}X^{--}(\tilde{u} + \hat{u}) \end{pmatrix}, \tag{4.35}$$

and likewise with suffix 2. Note that the symmetric 2×2 matrix $v^{\alpha\beta}$ still depends on three arbitrary real functions; as a consequence of Lorentz invariance, however, they are functions of ϕ and Y only. A similar explicit matrix representation may be given also for $F^{\pm\pm}{}_\alpha{}^\beta$.

4.2.2 Remaining Jacobi Identities

The Jacobi identities $J^{\phi BC} = 0$ have been taken care of automatically by the Lorentz covariant parametrization introduced in Section 4.2.1. In terms of these functions we write the remaining identities as

$$J^{\alpha\beta\gamma} = \vec{T}^{\alpha}(\mathcal{P}^{\beta\gamma}) + \mathrm{cycl}(\alpha\beta\gamma) = 0, \tag{4.36}$$

$$J^{\alpha\beta c} = \vec{T}^{c}(\mathcal{P}^{\alpha\beta}) + \vec{T}^{\alpha}(\chi F^c)^{\beta} + \vec{T}^{\beta}(\chi F^c)^{\alpha} = 0, \tag{4.37}$$

$$\frac{1}{2} J^{\alpha bc}\epsilon_{cb} = \vec{T}^{\alpha}(V) - \vec{T}^{b}(\chi F^c)^{\alpha}\epsilon_{cb} = 0. \tag{4.38}$$

Here $\vec{T}^a$ and $\vec{T}^\alpha$ are Hamiltonian vector fields introduced in (4.19), yielding ($\partial_\phi = \frac{\partial}{\partial\phi}$, $\partial_a = \frac{\partial}{\partial X^a}$, $\partial_\alpha = \frac{\partial}{\partial\chi^\alpha}$)

$$\vec{T}^{a} = X^b \epsilon_b{}^a \partial_\phi + \left(v + \frac{1}{2}\chi^2 v_2\right)\epsilon^{ab}\partial_b - (\chi F^a)^{\beta}\partial_\beta, \tag{4.39}$$

$$\vec{T}^{\alpha} = -\frac{1}{2}(\chi\gamma^3)^{\alpha}\partial_\phi + (\chi F^b)^{\alpha}\partial_b + \left(v^{\alpha\beta} + \frac{1}{2}\chi^2 v_2{}^{\alpha\beta}\right)\partial_\beta. \tag{4.40}$$

To find the solution of (4.36)–(4.38) it is necessary to expand in terms of the anticommutative coordinate χ^α. Therefore, it is convenient to split off any dependence on χ^α and its derivative also in (4.39) and (4.40), using instead the special Lorentz vector and spinor matrix valued derivatives[1]

$$\nabla^c := X^d \epsilon_d{}^c \partial_\phi + v\epsilon^{cd}\partial_d, \tag{4.41}$$

$$\nabla_\delta{}^\alpha := -\frac{1}{2}\gamma^3{}_\delta{}^\alpha\partial_\phi + F^d{}_\delta{}^\alpha\partial_d. \tag{4.42}$$

Then the Jacobi identities, arranged in the order $J^{\alpha\beta c}|$, $J^{\alpha\beta\gamma}|_\chi$, $J^{\alpha bc}|_\chi$ and $J^{\alpha\beta c}|_{\chi^2}$, that is the order of increasing complexity best adapted for our further analysis, read

$$v^{\alpha)\gamma}F^c{}_\gamma{}^{(\beta} + \frac{1}{2}\nabla^c v^{\alpha\beta} = 0, \tag{4.43}$$

$$v_\delta{}^\alpha v_2^{\beta\gamma} - \nabla_\delta{}^\alpha v^{\beta\gamma} + \mathrm{cycl}(\alpha\beta\gamma) = 0, \tag{4.44}$$

$$v_\delta{}^\alpha v_2 - \nabla_\delta{}^\alpha v + \nabla^c F^b{}_\delta{}^\alpha \epsilon_{bc} - (F^c F^b)_\delta{}^\alpha\epsilon_{bc} = 0, \tag{4.45}$$

$$\nabla^c v_2^{\alpha\beta} - F^c{}_\delta{}^\delta v_2^{\alpha\beta} + v_2\epsilon^{cd}\partial_d v^{\alpha\beta} + 2\nabla^{\delta(\alpha|}F^c{}_\delta{}^{|\beta)} + 2v_2^{\alpha)\delta}F^c{}_\delta{}^{(\beta} = 0. \tag{4.46}$$

All known solutions for $d = 2$ supergravity models found in the literature have the remarkable property that the Poisson tensor has (almost everywhere, i.e. except for isolated points) constant rank four, implying exactly one conserved Casimir function C [132]. Since the purely bosonic Poisson

[1]When (4.41) acts on an invariant function of ϕ and Y, ∇^c essentially reduces to the 'scalar' derivative, introduced in (4.22).

tensor has (almost everywhere) a maximum rank of two, this implies that the respective fermionic bracket $\mathcal{P}^{\alpha\beta}$ (or, equivalently, its χ-independent part $v^{\alpha\beta}$) must be of full rank if only one Casimir function is present in the fermionic extension. In the following subsection we will consider this case, i.e. we will restrict our attention to (regions in the target space with) invertible $\mathcal{P}^{\alpha\beta}$. For describing the rank we introduce the notation $(B|F)$. Here B denotes the rank of the bosonic body of the algebra, F the one of the extension. In this language the nondegenerate case has rank $(2|2)$. The remaining degenerate cases with rank $(2|0)$ and $(2|1)$ will be analyzed in a second step (Section 4.2.2 and Section 4.2.2).

Nondegenerate Fermionic Sector

When the matrix $v^{\alpha\beta}$ in (4.34) is nondegenerate, i.e. when its determinant

$$\Delta := \det(v^{\alpha\beta}) = \frac{1}{2} v^{\alpha\beta} v_{\beta\alpha} \tag{4.47}$$

is nonzero, for a given bosonic bracket this yields all supersymmetric extensions of maximal total rank. We note in parenthesis that due to the two-dimensionality of the spinor space (and the symmetry of $v^{\alpha\beta}$) the inverse matrix to $v^{\alpha\beta}$ is nothing else but $v_{\alpha\beta}/\Delta$, which is used in several intermediary steps below.

The starting point of our analysis of the remaining Jacobi identities $J^{ABC} = 0$ will always be a certain ansatz, usually for $v^{\alpha\beta}$. Therefore, it will be essential to proceed in a convenient sequence so as to obtain the restrictions on the remaining coefficient functions in the Poisson tensor with the least effort. This is also important because it turns out that several of these equations are redundant. This sequence has been anticipated in (4.43)–(4.46). There are already redundancies contained in the second and third step (eqs. (4.44) and (4.45)), while the χ^2-part of $J^{\alpha\beta c} = 0$ (eq. (4.46)) turns out to be satisfied identically because of the other equations. It should be noted, though, that this peculiar property of the Jacobi identities is *not* a general feature, resulting e.g. from some hidden symmetry, it holds true only in the case of a nondegenerate $\mathcal{P}^{\alpha\beta}$ (cf. the discussion of the degenerate cases below).

For fixed (nondegenerate) $v^{\alpha\beta}$, all solutions of (4.43) are parametrized by a Lorentz vector field f^a on the coordinate space (ϕ, X^a):

$$F^c{}_\alpha{}^\beta = \left[f^c \epsilon^{\gamma\beta} - \nabla^c v^{\gamma\beta} \right] \frac{v_{\gamma\alpha}}{2\Delta} \tag{4.48}$$

Eq. (4.44) can be solved to determine $v_2^{\alpha\beta}$ in terms of $v^{\alpha\beta}$:

$$v_2{}^{\alpha\beta} = -\frac{1}{4\Delta} v_\gamma{}^\delta \left[\nabla_\delta{}^\gamma v^{\alpha\beta} + \mathrm{cycl}(\alpha\beta\gamma) \right] \tag{4.49}$$

Multiplying (4.45) by $v_\beta{}^\gamma$ yields

$$\Delta\delta_\beta{}^\alpha v_2 = v_\beta{}^\delta \left[-\nabla_\delta{}^\alpha v + \nabla^c F^b{}_\delta{}^\alpha \epsilon_{bc} - (F^c F^b)_\delta{}^\alpha \epsilon_{bc} \right]. \tag{4.50}$$

The trace of (4.50) determines v_2, which is thus seen to depend also on the original bosonic potential v of (1.13).

Neither the vanishing traces of (4.50) multiplied with γ^3 or with γ_a, nor the identity (4.46) provide new restrictions in the present case. This has been checked by extensive computer calculations [142], based upon the explicit parametrization (4.25)–(4.33), which were necessary because of the extreme algebraic complexity of this problem. It is a remarkable feature of (4.48), (4.49) and (4.50) that the solution of the Jacobi identities for the nondegenerate case can be obtained from algebraic equations only.

As explained at the end of Section 4.1.2 the fermionic extension of the bosonic Casimir function c can be derived from $\{\chi^\alpha, C\} = 0$. The general result for the nondegenerate case we note here for later reference

$$c_2 = -\frac{1}{2\Delta} v_\alpha{}^\beta \left(-\frac{1}{2}\gamma^3{}_\beta{}^\alpha \partial_\phi + F^d{}_\beta{}^\alpha \partial_d \right) c. \tag{4.51}$$

The algebra of full rank (2|2) with the above solution for $F^c{}_\alpha{}^\beta$, $v_2{}^{\alpha\beta}$ and v_2 depends on 6 independent functions v, $v^{\alpha\beta}$ and f^a and their derivatives. The original bosonic model determines the 'potential' v in (1.2) or (1.13). Thus the arbitrariness of $v^{\alpha\beta}$ and f^a indicates that the supersymmetric extensions, obtained by fermionic extension from the PSM, are far from unique. This has been mentioned already in the previous section and we will further illuminate it in the following one.

Degenerate Fermionic Sector, Rank (2|0)

For vanishing rank of $\mathcal{P}^{\alpha\beta}|$, i.e. $v^{\alpha\beta} = 0$, the identities (4.43) and (4.44) hold trivially whereas the other Jacobi identities become complicated differential equations relating F^a, v and $v_2^{\alpha\beta}$. However, these equations can again be reduced to algebraic ones for these functions when the information on additional Casimir functions is employed, which appear in this case. These have to be of fermionic type with the general ansätze

$$C^{(+)} = \chi^+ \left| \frac{X^{--}}{X^{++}} \right|^{\frac{1}{4}} c_{(+)}, \tag{4.52}$$

$$C^{(-)} = \chi^- \left| \frac{X^{--}}{X^{++}} \right|^{-\frac{1}{4}} c_{(-)}. \tag{4.53}$$

The quotients X^{--}/X^{++} assure that $c_{(\pm)}$ are Lorentz invariant functions of ϕ and Y. This is possible because the Lorentz boosts in two dimensions do not mix chiral components and the light cone coordinates $X^{\pm\pm}$.

Taking a Lorentz covariant ansatz for the Poisson tensor as specified in Section 4.2.1, $C^{(+)}$ and $C^{(-)}$ must obey $\{X^a, C^{(+)}\} = \{X^a, C^{(-)}\} = 0$. Both expressions are linear in χ^α, therefore, the coefficients of χ^α have to vanish separately. This leads to $F^a{}_-{}^+ = 0$ and $F^a{}_+{}^- = 0$ immediately. With the chosen representation of the γ-matrices (cf. Appendix B) it is seen that (4.30) is restricted to $f^{ab} = 0$, i.e. the potentials $f_{(s)}$, $f_{(t)}$, $f_{(h)}$ and $f_{(a)}$ have to vanish. A further reduction of the system of equations reveals the further conditions $f_{(11)} = 0$[2] and

$$v = 4Y f_{(31)}. \tag{4.54}$$

This leaves the differential equations for $c_{(+)}$ and $c_{(-)}$

$$\left(\nabla + f_{(12)} + f_{(32)}\right) c_{(+)} = 0, \tag{4.55}$$

$$\left(\nabla + f_{(12)} - f_{(32)}\right) c_{(-)} = 0. \tag{4.56}$$

The brackets $\{\chi^+, C^{(+)}\}$ and $\{\chi^-, C^{(-)}\}$ are proportional to χ^2; the resulting equations require $\tilde{u}_2 = \hat{u}_2 = 0$. The only surviving term u_2 of $\mathcal{P}^{\alpha\beta}$ is related to F^a via $u_2 = -f_{(12)}$ as can be derived from $\{\chi^-, C^{(+)}\} = 0$ as well as from $\{\chi^+, C^{(-)}\} = 0$, which are equations of order χ^2 too.

Thus the existence of the fermionic Casimir functions (4.52) and (4.53) has lead us to a set of *algebraic* equations among the potentials of the Lorentz covariant ansatz for the Poisson tensor, and the number of independent potentials has been reduced drastically. The final question, whether the Jacobi identities are already fulfilled with the relations found so far finds a positive answer, and the general Poisson tensor with degenerate fermionic sector, depending on four parameter functions $v(\phi, Y)$, $v_2(\phi, Y)$, $f_{(12)}(\phi, Y)$ and $f_{(32)}(\phi, Y)$ reads

$$\mathcal{P}^{ab} = \left(v + \frac{1}{2}\chi^2 v_2\right) \epsilon^{ab}, \tag{4.57}$$

$$\mathcal{P}^{\alpha b} = \frac{v}{4Y} X^b (\chi\gamma^3)^\alpha - f_{(32)} X^c \epsilon_c{}^b (\chi\gamma^3)^\alpha - f_{(12)} X^c \epsilon_c{}^b \chi^\alpha, \tag{4.58}$$

$$\mathcal{P}^{\alpha\beta} = -\frac{1}{2}\chi^2 f_{(12)} \gamma^{3\alpha\beta}. \tag{4.59}$$

This Poisson tensor possesses three Casimir functions: two fermionic ones defined in regions $Y \neq 0$ according to (4.52) and (4.53), where $c_{(+)}$ and $c_{(-)}$ have to fulfill the first order differential equations (4.55) and (4.56), respectively, and one bosonic Casimir function C of the form (4.21), where c is a solution of the bosonic differential equation (4.22)—note the definition of ∇ therein—and where c_2 has to obey

$$\left(\nabla + 2f_{(12)}\right) c_2 = v_2 \partial_Y c. \tag{4.60}$$

[2]In fact $f_{(11)}$ vanishes in all cases, i.e. also for rank $(2|2)$ and $(2|1)$.

Let us finally emphasize that it was decisive within this subsection to use the information on the *existence* of Casimir functions. This follows from the property of the bivector $\mathcal{P}^{IJ}$ to be surface-forming, which in turn is a consequence of the (graded) Jacobi identity satisfied by the bivector. However, the inverse does not hold in general: Not any surface-forming bivector satisfies the Jacobi identities. Therefore, it was necessary to check their validity in a final step.

Degenerate Fermionic Sector, Rank (2|1)

When the fermionic sector has maximal rank one, again the existence of a fermionic Casimir function is very convenient. We start with 'positive chirality'[3]. We choose the ansatz (cf. Appendix B)

$$\mathcal{P}^{\alpha\beta}| = v^{\alpha\beta} = i\tilde{u}X^c(\gamma_c P_+)^{\alpha\beta} = \begin{pmatrix} \sqrt{2}\tilde{u}X^{++} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.61}$$

The most general case of rank (2|1) can be reduced to (4.61) by a (target space) transformation of the spinors. Negative chirality where P_+ is replaced with P_- is considered below. Testing the ansätze (4.52) and (4.53) reveals that $C^{(-)}$ now again is a Casimir function, but $C^{(+)}$ is not. Indeed $\{\chi^+, C^{(+)}\}| \propto \tilde{u}c_{(+)} \neq 0$ in general, whereas $\{\chi^+, C^{(-)}\}| \equiv 0$ shows that the fermionic Casimir function for positive chirality is $C^{(-)}$, where $c_{(-)} = c_{(-)}(\phi, Y)$ has to fulfill a certain differential equation, to be determined below.

The existence of $C^{(-)}$ can be used to obtain information about the unknown components of $\mathcal{P}^{AB}$. Indeed an investigation of $\{X^A, C^{(-)}\} = 0$ turns out to be much simpler than trying to get that information directly from the Jacobi identities. The bracket $\{X^a, C^{(-)}\} = 0$ results in ${F^a}_+{}^- = 0$ and from $\{\chi^\alpha, C^{(-)}\} = 0$ the relation ${v_2}^{--} = 0$ can be derived. This is the reason why the ansatz (4.30) and (4.33), retaining (4.31) and (4.32), attains the simpler form

$$F^a = f^a_{(1)}\mathbb{1} + if^{ab}(\gamma_b P_+) + f^a_{(3)}\gamma^3, \tag{4.62}$$

$$f^{ab} = f_{(s)}\eta^{ab} + f_{(t)}X^a X^b. \tag{4.63}$$

Likewise for the χ^2-component of $\mathcal{P}^{\alpha\beta}$ we set

$${v_2}^{\alpha\beta} = i\tilde{u}_2 X^c(\gamma_c P_+)^{\alpha\beta} + u_2\gamma^{3\alpha\beta}. \tag{4.64}$$

Not all information provided by the existence of $C^{(-)}$ has been introduced at this point. Indeed using the chiral ansatz (4.61) together with

[3]'Positive chirality' refers to the structure of (4.61). It does not preclude the coupling to the negative chirality component χ^- in other terms. A genuine chiral algebra (similar to $N = (1, 0)$ supergravity) is a special case to be discussed below in Section 4.6.6

(4.62)–(4.64) the calculation of $\{X^a, C^{(-)}\} = 0$ in conjunction with the Jacobi identities $J^{\alpha\beta c}| = 0$ (cf. (4.43)) requires $f_{(11)} = 0$ and

$$v = 4Y f_{(31)}. \tag{4.65}$$

It should be noted that the results $f_{(11)} = 0$ and (4.65) follow from $\{\chi^\alpha, C\} = 0$ too, where C is a bosonic Casimir function. The remaining equation in $\{X^a, C^{(-)}\} = 0$ together with $\{\chi^\alpha, C^{(-)}\} = 0$ yields $u_2 = -f_{(12)}$. With the solution obtained so far any calculation of $\{X^A, C^{(-)}\} = 0$ leads to one and only one differential equation (4.56) which must be satisfied in order that (4.53) is a Casimir function.

We now turn our attention to the Jacobi identities. The inspection of $J^{++c}| = 0$ (4.43), $J^{+++}|_\chi = 0$ (4.44) and $J^{+bc}|_\chi = 0$ (4.45) leads to the conditions

$$f_{(32)} = \frac{1}{2}(\nabla \ln |\tilde{u}|) - f_{(12)} - \frac{v}{4Y}, \tag{4.66}$$

$$\tilde{u}_2 = f(\partial_Y \ln |\tilde{u}|) + f_{(t)}, \tag{4.67}$$

$$v_2 = \big(\nabla + 2f_{(12)} + (\partial_Y v)\big)\frac{f}{\tilde{u}}, \tag{4.68}$$

respectively. In order to simplify the notation we introduced

$$f = f_{(s)} + 2Y f_{(t)}. \tag{4.69}$$

All other components of the Jacobi tensor are found to vanish identically.

The construction of graded Poisson tensors with 'negative chirality', i.e. with fermionic sector of the form

$$\mathcal{P}^{\alpha\beta}| = v^{\alpha\beta} = i\tilde{u}X^c(\gamma_c P_-)^{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\tilde{u}X^{--} \end{pmatrix}, \tag{4.70}$$

proceeds by the same steps as for positive chirality. Of course, the relevant fermionic Casimir function is now $C^{(+)}$ of (4.52) and P_+ in (4.62) and (4.64) has to be replaced by P_-. The results $f_{(11)} = 0$, (4.65), (4.67) and (4.68) remain the same, only $f_{(32)}$ acquires an overall minus sign,

$$f_{(32)} = -\frac{1}{2}(\nabla \ln |\tilde{u}|) + f_{(12)} + \frac{v}{4Y}, \tag{4.71}$$

to be inserted in the differential equation (4.55) for $c_{(+)}$.

The results for graded Poisson tensors of both chiralities can be summarized as (cf. (4.69))

$$\mathcal{P}^{ab} = \left(v + \frac{1}{2}\chi^2\left[\nabla + 2f_{(12)} + (\partial_Y v)\right]\frac{f}{\tilde{u}}\right)\epsilon^{ab}, \tag{4.72}$$

$$\mathcal{P}^{\alpha b} = (\chi F^b)^\alpha \tag{4.73}$$

$$\mathcal{P}^{\alpha\beta} = i\left(\tilde{u} + \frac{1}{2}\chi^2\left[f(\partial_Y \ln |\tilde{u}|) + f_{(t)}\right]\right)X^c(\gamma_c P_\pm)^{\alpha\beta} - \frac{1}{2}\chi^2 f_{(12)}\gamma^{3\alpha\beta}. \tag{4.74}$$

Eq. (4.73) reads explicitly

$$F^b = \frac{v}{4Y}(X^b \pm X^c \epsilon_c{}^b)\gamma^3 - 2f_{(12)} X^c \epsilon_c{}^b P_\mp \\ + i f_{(s)}(\gamma^b P_\pm) + i f_{(t)} X^b X^c (\gamma_c P_\pm) \mp \frac{1}{2}(\nabla \ln |\tilde{u}|) X^c \epsilon_c{}^b \gamma^3. \quad (4.75)$$

Eqs. (4.72)–(4.75) represent the generic solution of the graded $N = 1$ Poisson algebra of rank (2|1). In addition to $v(\gamma, Y)$ it depends on four parameter functions $\tilde{u}$, $f_{(12)}$, $f_{(s)}$ and $f_{(t)}$, all depending on ϕ and Y.

Each chiral type possesses a bosonic Casimir function $C = c + \frac{1}{2}\chi^2 c_2$, where $c(\phi, Y)$ and $c_2(\phi, Y)$ are determined by $\nabla c = 0$ and

$$c_2 = \frac{f \partial_Y c}{\tilde{u}}. \quad (4.76)$$

The fermionic Casimir function for positive chirality is $C^{(-)}$ and for negative chirality $C^{(+)}$ (cf. (4.53) and (4.52)), where $c_{(\mp)}(\phi, Y)$ are bosonic scalar functions solving the same differential equation in both cases when eliminating $f_{(32)}$,

$$\left(\nabla + 2f_{(12)} + \frac{v}{4Y} - \frac{1}{2}(\nabla \ln |\tilde{u}|)\right) c_{(\mp)} = 0, \quad (4.77)$$

derived from (4.56) with (4.66) and from (4.55) with (4.71).

4.3 Target space diffeomorphisms

When subjecting the Poisson tensor of the action (4.12) to a diffeomorphism

$$X^I \to \bar{X}^I = \bar{X}^I(X) \quad (4.78)$$

on the target space $\mathcal{N}$, another action of gPSM form is generated with the new Poisson tensor

$$\bar{\mathcal{P}}^{IJ} = (\bar{X}^I \overleftarrow{\partial}_K)\mathcal{P}^{KL}(\vec{\partial}_L \bar{X}^J). \quad (4.79)$$

It must be emphasized that in this manner a *different* model is created with—in the case of 2d gravity theories and their fermionic extensions—in general different bosonic 'body' (and global topology). Therefore, such transformations are a powerful tool to create new models from available ones. This is important, because—as shown in Section 4.2 above—the solution of the Jacobi identities as a rule represents a formidable computational problem. This problem could be circumvented by starting from a simple $\bar{\mathcal{P}}^{IJ}(\bar{X})$, whose Jacobi identities have been solved rather trivially. As a next

step a transformation (4.78) is applied. The most general Poisson tensor can be generated by calculating the inverse of the Jacobi matrices

$$J_I{}^{\bar{J}}(X) = \vec{\partial}_I \bar{X}^J, \qquad J_I{}^{\bar{K}}(J^{-1})_{\bar{K}}{}^J = \delta_I{}^J, \tag{4.80}$$

$$I^{\bar{I}}{}_J(X) = \bar{X}^I \overleftarrow{\partial}_J, \qquad (I^{-1})^I{}_{\bar{K}} I^{\bar{K}}{}_J = \delta_I{}^J. \tag{4.81}$$

According to

$$P^{IJ}(X) = (I^{-1})^I{}_{\bar{K}} \bar{P}^{KL}|_{\bar{X}(X)} (J^{-1})_{\bar{L}}{}^J \tag{4.82}$$

the components $\mathcal{P}^{IJ}$ of the transformed Poisson tensor are expressed in terms of the coordinates X^I without the need to invert (4.78).

The drawback of this argument comes from the fact that in our problem the (bosonic) part of the 'final' algebra is given, and the inverted version of the procedure described here turns out to be very difficult to implement.

Nevertheless, we construct explicitly the diffeomorphisms connecting the dilaton prepotential superalgebra given in Section 4.4.4 with a prototype Poisson tensor in its simplest form, i.e. with a Poisson tensor with constant components. Coordinates where the nonzero components take the values ± 1 are called Casimir-Darboux coordinates. This immediately provides the explicit solution of the corresponding gPSM too; for details cf. Section 4.7.

In addition, we have found target space diffeomorphisms very useful to incorporate e.g. bosonic models related by conformal transformations. An example of that will be given in Section 4.4.5 where an algebra referring to models without bosonic torsion—the just mentioned dilaton prepotential algebra—can be transformed quite simply to one depending quadratically on torsion and thus representing a dilaton theory with kinetic term ($Z \neq 0$ in (1.3)) in its dilaton version. There the identification $A_I = (\omega, e_a, \psi_\alpha)$ with 'physical' Cartan variables is used to determine the solution of the latter theory ($Z \neq 0$) from the simpler model ($\bar{Z} = 0$) with PSM variables $(\bar{X}^I, \bar{A}_I)$ by

$$A_I = (\vec{\partial}_I \bar{X}^J) \bar{A}_J. \tag{4.83}$$

Interesting information regarding the arbitrariness to obtain supersymmetric extensions of bosonic models as found in the general solutions of Section 4.2 can also be collected from target space diffeomorphisms. Imagine that a certain gPSM has been found, solving the Jacobi identities with a particular ansatz. A natural question would be to find out which other models have the same bosonic body. For this purpose we single out at first all transformations (4.78) which leave the components $\mathcal{P}^{A\phi}$ form invariant as given by (1.12) and (4.1):

$$\bar{\phi} = \phi, \qquad \bar{X}^a = X^b C_b{}^a, \qquad \bar{\chi}^\alpha = \chi^\beta h_\beta{}^\alpha. \tag{4.84}$$

Here $C_b{}^a$ and $h_\beta{}^\alpha$ are Lorentz covariant functions (resp. spinor matrices)

$$C_b{}^a = L\delta_b{}^a + M\epsilon_b{}^a = c_b{}^a + \frac{1}{2}\chi^2 (c_2)_b{}^a, \tag{4.85}$$

$$h_\beta{}^\alpha = \Big[h_{(1)} \mathbb{1} + h_{(2)}\gamma^3 + ih_{(3)} X^c \gamma_c + ih_{(4)} X^d \epsilon_d{}^c \gamma_c \Big]_\beta{}^\alpha, \tag{4.86}$$

when expressed in terms of χ^2 ($L = l + \frac{1}{2}\chi^2 l_2$ and similar for M) and in terms of ϕ and Y (l, l_2, m, m_2, $h_{(i)}$).

The 'stabilisator' ($\bar{\mathcal{P}}^{ab} = \mathcal{P}^{ab}$) of the bosonic component $v(\phi, Y) = v(\bar{\phi}, \bar{Y})$ of a graded Poisson tensor will be given by the restriction of $c_b{}^a$ to a Lorentz transformation on the target space $\mathcal{N}$ with $l^2 - m^2 = 1$ in (4.85). Furthermore from the two parameters $h_{(1)}$ and $h_{(2)}$ a Lorentz transformation can be used to reduce them to one independent parameter. Thus no less than five arbitrary two argument functions are found to keep the bosonic part of $\mathcal{P}^{ab}$ unchanged, but produce different fermionic extensions with supersymmetries different from the algebra we started from. This number for rank (2|2) exactly coincides with the number of arbitrary invariant functions found in Section 4.2.2. For rank (2|1) in the degenerate case a certain 'chiral' combination of $h_{(3)}$ and $h_{(4)}$ in (4.86) must be kept fixed, reducing that number to four—again in agreement with Section 4.2.2. In a similar way also the appearance of just three arbitrary functions in Section 4.2.2 for rank (2|0) can be understood.

4.4 Particular Poisson Superalgebras

The compact formulae of the last sections do not seem suitable for a general discussion, especially in view of the large arbitrariness of gPSMs. We, therefore, elucidate the main features in special models of increasing complexity. The corresponding actions and their relations to the alternative formulations (1.1) and, or the dilaton theory form (1.3) will be discussed in Section 4.6.

4.4.1 Block Diagonal Algebra

The most simple ansatz which, nevertheless, already shows the generic features appearing in fermionic extensions, consists in setting the mixed components $\mathcal{P}^{\alpha b} = 0$ so that the nontrivial fermionic brackets are restricted to the block $\mathcal{P}^{\alpha\beta}$. Then (4.43)–(4.46) reduce to

$$\nabla^c v^{\alpha\beta} = 0, \tag{4.87}$$

$$v_\delta{}^\alpha v_2{}^{\beta\gamma} + \frac{1}{2}\gamma^3{}_\delta{}^\alpha \partial_\phi v^{\beta\gamma} + \mathrm{cycl}(\alpha\beta\gamma) = 0, \tag{4.88}$$

$$v_\delta{}^\alpha v_2 + \frac{1}{2}\gamma^3{}_\delta{}^\alpha \partial_\phi v = 0, \tag{4.89}$$

$$v_2 \epsilon^{cd} \partial_d v^{\alpha\beta} + \nabla^c v_2^{\alpha\beta} = 0. \tag{4.90}$$

Eq. (4.89) implies the spinorial structure

$$v^{\alpha\beta} = u\gamma^{3\alpha\beta}. \tag{4.91}$$

The trace of (4.87) with γ^3 leads to the condition (4.22) for u, i.e. $u = u(c(\phi, Y))$ depends on the combination of ϕ and Y as determined by the bosonic Casimir function.

For $u \neq 0$ the remaining equations (4.88), (4.89) and (4.90) are fulfilled by

$$v_2^{\alpha\beta} = -\frac{\partial_\phi u}{2u}\gamma^{3\alpha\beta}, \qquad v_2 = -\frac{\partial_\phi v}{2u}. \tag{4.92}$$

For the present case according to (4.51) the Casimir function is

$$C = c - \frac{1}{2}\chi^2 \frac{\partial_\phi c}{2u(c)}. \tag{4.93}$$

It is verified easily that

$$U = u(C) = u(c) + \frac{1}{2}\chi^2 u_2, \qquad u_2 = -\frac{\partial_\phi c}{2u(c)}\frac{du}{dc}. \tag{4.94}$$

Already in this case we observe that in the fermionic extension Δ^{-1}, u^{-1} from the inverse of $v^{\alpha\beta}$ may introduce singularities. It should be emphasized that $u = u(c)$ is an arbitrary function of $c(\phi, Y)$. Except for $u = u_0 = \text{const}$ (see below) any generic choice of the arbitrary function $u(c)$ by the factors u^{-1} in (4.92), thus may introduce restrictions on the allowed range of ϕ and Y or new singularities on a certain surface where $u(c(\phi, Y))$ vanishes, not present in the purely bosonic bracket. Indeed, these obstructions in certain fermionic extensions are a generic feature of gPSMs. The singularities are seen to be caused here by Δ^{-1}, the inverse of the determinant (4.47), except for cases with $\Delta = \text{const}$ or when special cancellation mechanisms are invoked. Another source for the same phenomenon will appear below in connection with the appearance of a 'prepotential' for v. Still, such 'obstructions' can be argued to be rather harmless. We will come back to these issues in several examples below, especially when discussing an explicit solution in Section 4.7.

This complication can be made to disappear by choosing $u = u_0 = \text{const} \neq 0$. Then the fermionic extension ($v' = \partial_\phi v$)

$$\mathcal{P}^{ab} = \left(v - \frac{1}{4u_0}\chi^2 v'\right)\epsilon^{ab}, \tag{4.95}$$

$$\mathcal{P}^{\alpha b} = 0, \tag{4.96}$$

$$\mathcal{P}^{\alpha\beta} = u_0\gamma^{3\alpha\beta} \tag{4.97}$$

does not lead to restrictions on the purely bosonic part $v(\phi, Y)$ of the Poisson tensor, nor does it introduce additional singularities, besides the ones which may already be present in the potential $v(\phi, Y)$. But then no genuine supersymmetry survives (see Section 4.6.1 below).

It should be noted that all dilaton models mentioned in the introduction can be accommodated in a nontrivial version $u \neq$ const of this gPSM. We shall call the corresponding supergravity actions their 'diagonal extensions'.

4.4.2 Nondegenerate Chiral Algebra

Two further models follow by setting $u = u_0 = \text{const} \neq 0$ and $\hat{u} = \pm\tilde{u}_0 = \text{const}$. In this way a generalization with full rank of the chiral $N = (1,0)$ and $N = (0,1)$ algebras is obtained (cf. Appendix B)

$$v^{\alpha\beta} = i\tilde{u}_0 X^c (\gamma_c P_\pm)^{\alpha\beta} + u_0 \gamma^{3\alpha\beta}. \tag{4.98}$$

This particular choice for the coefficients of X^c in $v^{\alpha\beta}$ also has the advantage that X^c drops out from $\Delta = -u_0^2/4$, thus its inverse exists everywhere. Restricting furthermore $f^c = 0$ we arrive at

$$F^c{}_\alpha{}^\beta = -\frac{i\tilde{u}_0 v}{2u_0}(\gamma^c P_\pm)_\alpha{}^\beta, \qquad v_2{}^{\alpha\beta} = 0, \qquad v_2 = -\frac{v'}{2u_0}. \tag{4.99}$$

This yields another graded Poisson tensor for the arbitrary bosonic potential $v(\phi, Y)$

$$\mathcal{P}^{ab} = \left(v - \frac{1}{4u_0}\chi^2 v'\right)\epsilon^{ab}, \tag{4.100}$$

$$\mathcal{P}^{\alpha b} = -\frac{i\tilde{u}_0 v}{2u_0}(\chi\gamma^b P_\pm)^\alpha, \tag{4.101}$$

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}_0 X^c (\gamma_c P_\pm)^{\alpha\beta} + u_0\gamma^{3\alpha\beta}. \tag{4.102}$$

There also are no obstructions for such models corresponding to any bosonic gravity model, given by a particular choice of $v(\phi, Y)$.

The Casimir function (cf. (4.51)) reads

$$C = c - \frac{1}{4u_0}\chi^2 c', \tag{4.103}$$

where c must obey (4.22).

4.4.3 Deformed Rigid Supersymmetry

The structure of rigid supersymmetry is encoded within the Poisson tensor by means of the components $v = 0$ and (cf. (4.35))

$$v^{\alpha\beta} = i\tilde{u}_0 X^c \gamma_c{}^{\alpha\beta} = \begin{pmatrix} \sqrt{2}\tilde{u}_0 X^{++} & 0 \\ 0 & \sqrt{2}\tilde{u}_0 X^{--} \end{pmatrix}, \tag{4.104}$$

where again $\tilde{u} = \tilde{u}_0 = \text{const} \neq 0$. Here $\Delta = 2Y\tilde{u}_0^2$ and

$$\frac{1}{\Delta}v_{\alpha\beta} = \frac{i}{2Y\tilde{u}_0}X^c\gamma_{c\alpha\beta} = \begin{pmatrix} \frac{1}{\sqrt{2}\tilde{u}_0X^{++}} & 0 \\ 0 & \frac{1}{\sqrt{2}\tilde{u}_0X^{--}} \end{pmatrix}. \tag{4.105}$$

Generalizing this ansatz to $v \neq 0$, the simplest choice $f^c = 0$ with an arbitrary function $v(\phi, Y)$ (deformed rigid supersymmetry, DRS) yields

$$F^c{}_\alpha{}^\beta = \frac{v}{4Y}X^a(\gamma_a\gamma^c\gamma^3)_\alpha{}^\beta, \qquad v_2{}^{\alpha\beta} = \frac{v}{4Y}\gamma^{3\alpha\beta}, \qquad v_2 = 0, \tag{4.106}$$

and thus for $\mathcal{P}^{IJ}$

$$\mathcal{P}^{ab} = v\epsilon^{ab}, \tag{4.107}$$

$$\mathcal{P}^{\alpha b} = \frac{v}{4Y}X^c(\chi\gamma_c\gamma^b\gamma^3)^\alpha, \tag{4.108}$$

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}_0X^c\gamma_c{}^{\alpha\beta} + \frac{1}{2}\chi^2\frac{v}{4Y}\gamma^{3\alpha\beta}, \tag{4.109}$$

and for the Casimir function $C = c$ with (4.22).

From (4.107)–(4.109) it is clear—in contrast to the algebras 4.4.1 and 4.4.2—that this fermionic extension for a generic $v \neq 0$ introduces a possible further singularity at $Y = 0$, which cannot be cured by further assumptions on functions which are still arbitrary.

Of course, in order to describe flat spacetime, corresponding to the Poisson tensor of rigid supersymmetry, one has to set $v(\phi, Y) = 0$. Then the singularity at $Y = 0$ in the extended Poisson tensor disappears.

We remark already here that despite the fact that for $v \neq 0$ the corresponding supersymmetrically extended action functional (in contrast to its purely bosonic part) becomes singular at field values $Y \equiv \frac{1}{2}X^aX_a = 0$, we expect that if solutions of the field equations are singular there as well, such singularities will not be relevant if suitable 'physical' observables are considered. We have in mind the analogy to curvature invariants which are not affected by 'coordinate singularities'. We do, however, not intend to prove this statement in detail within the present thesis; in Section 4.7 below we shall only shortly discuss the similar singularities, caused by the prepotential in an explicit solution of the related field-theoretical model.

4.4.4 Dilaton Prepotential Algebra

We now assume that the bosonic potential v is restricted to be a function of the dilaton ϕ only, $\dot{v} = \partial_Y v = 0$. Many models of 2d supergravity, already known in the literature, are contained within algebras of this type, one of which was described in ref. [129–132]. Let deformed rigid supersymmetry of Section 4.4.3 again be the key component of the Poisson tensor (4.104). Our attempt in Section 4.4.3 to provide a Poisson tensor for arbitrary v built

around that component produced a new singularity at $Y = 0$ in the fermionic extension. However, the Poisson tensor underlying the model considered in [129–132] was not singular in Y. Indeed there exists a mechanism by which this singularity can be cancelled in the general solution (4.47)–(4.50), provided the arbitrary functions are chosen in a specific manner.

For this purpose we add to (4.104), keeping $\tilde{u} = \tilde{u}_0 = \text{const}$, the fermionic potential $u(\phi)$,

$$v^{\alpha\beta} = i\tilde{u}_0 X^c {\gamma_c}^{\alpha\beta} + u\gamma^{3\alpha\beta} = \begin{pmatrix} \sqrt{2}\tilde{u}_0 X^{++} & -u \\ -u & \sqrt{2}\tilde{u}_0 X^{--} \end{pmatrix}, \tag{4.110}$$

with determinant

$$\Delta = 2Y\tilde{u}_0^2 - u^2. \tag{4.111}$$

The Hamiltonian vector field $\vec{T}^c$ in the solution (4.48) generates a factor $f_{(t)} \neq 0$ in (4.33). The independent vector field f^c can be used to cancel that factor provided one chooses

$$f^c = \frac{1}{2}u' X^c. \tag{4.112}$$

Then the disappearance of $f_{(t)}$ is in agreement with the solution given in ref. [131]. The remaining coefficient functions then follow as

$$F^c{}_\alpha{}^\beta = \frac{1}{2\Delta}\left(\tilde{u}_0^2 v + uu'\right) X^a (\gamma_a \gamma^c \gamma^3)_\alpha{}^\beta + \frac{i\tilde{u}_0}{2\Delta}\left(uv + 2Yu'\right) \gamma^c{}_\alpha{}^\beta, \tag{4.113}$$

$$v_2{}^{\alpha\beta} = \frac{1}{2\Delta}\left(\tilde{u}_0^2 v + uu'\right) \gamma^{3\alpha\beta}, \tag{4.114}$$

$$v_2 = \frac{uv}{2\Delta^2}\left(\tilde{u}_0^2 v + uu'\right) + \frac{uu'}{2\Delta^2}\left(uv + 2Yu'\right) + \frac{1}{2\Delta}\left(uv' + 2Yu'\dot{v} + 2Yu''\right). \tag{4.115}$$

Up to this point the bosonic potential v and the potential u have been arbitrary functions of ϕ. Demanding now that

$$\tilde{u}_0^2 v + uu' = 0, \tag{4.116}$$

the singularity at $\Delta = 0$ is found to be cancelled not only in the respective first terms of (4.113)–(4.115), but also in the rest:

$$v = -\frac{(u^2)'}{2\tilde{u}_0^2}, \qquad F^c{}_\alpha{}^\beta = \frac{iu'}{2\tilde{u}_0}\gamma^c{}_\alpha{}^\beta, \tag{4.117}$$

$$v_2{}^{\alpha\beta} = 0, \qquad v_2 = \frac{u''}{2\tilde{u}_0^2}. \tag{4.118}$$

Furthermore the fermionic potential $u(\phi)$ is seen to be promoted to a 'prepotential' for $v(\phi)$. A closer look at (4.116) with (4.111) shows that this

relation is equivalent to $\nabla\Delta = 0$ which happens to be precisely the defining equation (4.22) of the Casimir function $c(\phi, Y)$ of the bosonic model. The complete Casimir function follows from (4.51):

$$c_2 = \frac{1}{2\Delta}\left(u\partial_\phi + 2Yu'\partial_Y\right)c \qquad (4.119)$$

so that

$$C = \Delta + \frac{1}{2}\chi^2 u'. \qquad (4.120)$$

Thus the Poisson tensor for $v = v(\phi)$, related to $u(\phi)$ by (4.116), becomes

$$\mathcal{P}^{ab} = \frac{1}{2\tilde{u}_0^2}\left(-(u^2)' + \frac{1}{2}\chi^2 u''\right)\epsilon^{ab}, \qquad (4.121)$$

$$\mathcal{P}^{\alpha b} = \frac{iu'}{2\tilde{u}_0}(\chi\gamma^b)^\alpha, \qquad (4.122)$$

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}_0 X^c {\gamma_c}^{\alpha\beta} + u\gamma^{3\alpha\beta}, \qquad (4.123)$$

which is indeed free from singularities produced by the supersymmetric extension. However, this does not eliminate all pitfalls: Given a bosonic model described by a particular potential $v(\phi)$ where ϕ is assumed to take values in the interval $I \subseteq \mathbb{R}$, we have to solve (4.116) for the prepotential $u(\phi)$, i.e. the quadratic equation

$$u^2 = -2\tilde{u}_0^2 \int_{\phi_0}^{\phi} v(\varphi)d\varphi, \qquad (4.124)$$

which may possess a solution within the real numbers only for a restricted range $\phi \in J$. The interval J may have a nontrivial intersection with I or even none at all. Clearly no restrictions occur if v contains a potential for the dilaton which happens to lead to a negative definite integral on the r.h.s. of (4.124) for *all* values of ϕ in I (this happens e.g. if v contains only odd powers of ϕ with negative prefactors). On the other hand, the domain of ϕ is always restricted if v contains even potentials, as becomes immediately clear when viewing the special solutions given in Table 4.1. There the different potentials $v(\phi)$ are labelled according to the models: The string model with Λ = const of [85, 145, 146, 75, 97–104], JT is the Jackiw-Teitelboim model (1.8), $\overline{\text{SRG}}$ the spherically reduced black hole (1.6) in the conformal description (cf. Section 1); the cubic potential appeared in [46], R^2 gravity is self-explaining. Note that in the case of $\overline{\text{SRG}}$ $I = J = \mathbb{R}_+$ ($\phi > 0$), there is already a (harmless) restriction on allowed values of ϕ at the purely bosonic level, cf. (1.6).

So, as argued above, one may get rid of the singularities at $Y = 0$ of supersymmetric extensions obtained in the previous section. In some

Model	$v(\phi) = -\frac{(u^2)'}{2\tilde{u}_0^2}$	$u(\phi)$
	0	$\tilde{u}_0\lambda$
String	$-\Lambda$	$\pm\tilde{u}_0\sqrt{2\Lambda(\phi-\phi_0)}$
JT	$-\lambda^2\phi$	$\tilde{u}_0\lambda\phi$
R^2	$-\frac{\alpha}{2}\phi^2$	$\pm\tilde{u}_0\sqrt{\frac{\alpha}{3}(\phi^3-\phi_0^3)}$
Howe	$-2\lambda^2\phi^3$	$\tilde{u}_0\lambda\phi^2$
S$\bar{\text{R}}$G	$-\frac{\lambda^2}{\sqrt{\phi}}$	$2\tilde{u}_0\lambda\sqrt[4]{\phi}$

Table 4.1: Special Dilaton Prepotential Algebras

cases, however, this leads to a restricted range for allowed values of the dilaton, or, alternatively, to complex valued Poisson tensors. Similarly to our expectation of the harmlessness of the above mentioned $1/Y$-singularities on the level of the solutions (cf. also [132]), we expect that also complex-valued Poisson tensors are no serious obstacle (both of these remarks apply to the classical analysis only!). In fact, a similar scenario was seen to be harmless (classically) also in the Poisson Sigma formulation of the G/G model for compact gauge groups like $SU(2)$, cf. [106, 147]. We further illustrate these remarks for the class of supergravity models considered in [129–132] at the end of Section 4.7.

4.4.5 Bosonic Potential Linear in Y

In order to retain the Y-dependence and thus an algebra with bosonic torsion, we take solution (4.110)–(4.115) but instead of (4.116) we may also choose

$$v = -\frac{(u^2)'}{2\tilde{u}_0^2} - \frac{\Delta}{2}f, \tag{4.125}$$

where f is an arbitrary function of ϕ and Y. Thanks to the factor Δ also in this case the fermionic extension does not introduce new singularities at $\Delta = 0$.[4] Even if f is a function of ϕ only ($\dot{f} = 0$), this model is quadratic in (bosonic) torsion, because of (4.111). A straightforward calculation using

[4]Clearly also in (4.125) the replacement $\Delta f \to G(\Delta, \phi, Y)$ with $G(\Delta, \phi, Y)/\Delta$ regular at $\Delta = 0$ has a similar effect. But linearity in Δ is sufficient for our purposes.

(4.125) gives

$$F^c{}_\alpha{}^\beta = -\frac{\tilde{u}_0^2 f}{4} X^a (\gamma_a \gamma^c \gamma^3)_\alpha{}^\beta + i \left(\frac{u'}{2\tilde{u}_0} - \frac{\tilde{u}_0 u f}{4} \right) \gamma^c{}_\alpha{}^\beta, \tag{4.126}$$

$$v_2{}^{\alpha\beta} = -\frac{\tilde{u}_0^2 f}{4} \gamma^{3\alpha\beta}, \tag{4.127}$$

$$v_2 = \frac{1}{2} \left(\frac{u''}{\tilde{u}_0^2} - u'f - \frac{uf'}{2} + \frac{\tilde{u}_0^2 u f^2}{4} - \frac{2Yu'\dot{f}}{2} \right). \tag{4.128}$$

It is worthwhile to note that the present algebra, where the bosonic potential v is of the type (1.4), can be reached from the algebra of Section 4.4.4 with $\bar{v} = \bar{v}(\bar{\phi})$ by a conformal transformation, i.e. a target space diffeomorphism in the sense of Section 4.3. We use bars to denote quantities and potentials of the algebra of Section 4.4.4, but not for $\tilde{u}_0$ because it remains unchanged, i.e. $\bar{v} = -\frac{(\bar{u}^2)'}{2\tilde{u}_0^2}$. By

$$\phi = \bar{\phi}, \qquad X^a = e^{\varphi(\phi)} \bar{X}^a, \qquad \chi^\alpha = e^{\frac{1}{2}\varphi(\phi)} \bar{\chi}^\alpha, \tag{4.129}$$

the transformed Poisson tensor, expanded in terms of unbarred coefficient functions (cf. Section 4.2.1) becomes

$$\tilde{u} = \tilde{u}_0, \qquad \tilde{u}_2 = 0, \tag{4.130}$$

$$u = e^\varphi \bar{u}, \qquad u_2 = -\frac{1}{2}\varphi', \tag{4.131}$$

$$v = e^{2\varphi}\bar{v} - 2Y\varphi', \qquad v_2 = e^\varphi \frac{\bar{u}''}{2\tilde{u}_0^2}, \tag{4.132}$$

$$f_{(12)} = \frac{1}{2}\varphi', \qquad f_{(31)} = -\frac{1}{2}\varphi', \tag{4.133}$$

$$f_{(s)} = e^\varphi \frac{\bar{u}'}{2\tilde{u}_0} \tag{4.134}$$

and $f_{(11)} = f_{(32)} = f_{(t)} = f_{(h)} = 0$. When $u(\phi)$ and $\varphi(\phi)$ are taken as basic independent potentials we arrive at

$$v = -\frac{1}{2\tilde{u}_0^2} e^{2\varphi} \left(e^{-2\varphi} u^2 \right)' - 2Y\varphi', \tag{4.135}$$

$$v_2 = \frac{1}{2\tilde{u}_0^2} e^{\varphi} \left(e^{-\varphi} u \right)'', \tag{4.136}$$

$$f_{(s)} = \frac{1}{2\tilde{u}_0} e^{\varphi} \left(e^{-\varphi} u \right)'. \tag{4.137}$$

If we set $\varphi' = \tilde{u}_0^2 f/2$ we again obtain solution (4.125)–(4.128) for Y-independent f. The components $\bar{\mathcal{P}}^{a\phi}$ and $\bar{\mathcal{P}}^{\alpha\phi}$ remain form invariant,

$$\mathcal{P}^{a\phi} = X^b \epsilon_b{}^a, \qquad \mathcal{P}^{\alpha\phi} = -\frac{1}{2}\chi^\beta \gamma^3{}_\beta{}^\alpha, \tag{4.138}$$

in agreement with the requirement determined for this case in Section 4.2.1. For completeness we list the transformation of the 1-forms $A_I = (\omega, e_a, \psi_\alpha)$ according to (4.83)

$$\omega = \bar{\omega} - \varphi' \left(\bar{X}^b \bar{e}_b + \frac{1}{2} \bar{\chi}^\beta \bar{\psi}_\beta \right), \qquad e_a = e^{-\varphi} \bar{e}_a, \qquad \psi_\alpha = e^{-\frac{1}{2}\varphi} \bar{\psi}_\alpha. \tag{4.139}$$

The second equation in (4.139) provide the justification for the name 'conformal transformation'.

With the help of the scaling parameter φ we can write (4.135), and also (4.125), in its equivalent form $\nabla(e^{-2\varphi}\Delta) = 0$, thus exposing the Casimir function to be $c(\phi, Y) = e^{-2\varphi}\Delta$. Now $u(\phi)$ and $\varphi(\phi)$ are to be viewed as two independent parameter functions labelling specific types of Poisson tensors. The solution

$$\mathcal{P}^{ab} = \left(-\frac{1}{2\tilde{u}_0^2} e^{2\varphi} \left(e^{-2\varphi} u^2 \right)' - 2Y\varphi' + \frac{1}{4\tilde{u}_0^2} \chi^2 e^{\varphi} \left(e^{-\varphi} u \right)'' \right) \epsilon^{ab}, \tag{4.140}$$

$$\mathcal{P}^{\alpha b} = -\frac{1}{2} \varphi' X^a (\chi \gamma_a \gamma^b \gamma^3)^\alpha + \frac{i}{2\tilde{u}_0} e^{\varphi} \left(e^{-\varphi} u \right)' (\chi \gamma^b)^\alpha, \tag{4.141}$$

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}_0 X^c {\gamma_c}^{\alpha\beta} + \left(u - \frac{1}{4} \chi^2 \varphi' \right) \gamma^{3\alpha\beta} \tag{4.142}$$

does not introduce a new singularity at $Y = 0$, but in order to provide the extension of the bosonic potential (1.4) we have to solve (4.135) for the scaling parameter $\varphi(\phi)$ and the fermionic potential $u(\phi)$, which may again lead to obstructions similar to the ones described at the end of Section 4.4.4. With the integrals over $Z(\phi)$ and $V(\phi)$ introduced in (4.24) we find

$$\varphi = -\frac{1}{2} Q(\phi), \tag{4.143}$$

$$u = \pm \sqrt{-2\tilde{u}_0^2 e^{-Q(\phi)} W(\phi)}. \tag{4.144}$$

Now we can read off the restriction to be $W(\phi) < 0$, yielding singularities at the boundary $W(\phi) = 0$. The ansatz (4.125) can be rewritten in the equivalent form

$$\nabla(e^Q \Delta) = 0 \Leftrightarrow c(\phi, Y) = e^Q \Delta = 2\tilde{u}_0^2 (Y e^Q + W). \tag{4.145}$$

The complete Casimir from (4.51), which again exhibits the simpler form (4.119), reads

$$C = e^Q \left(\Delta + \frac{1}{2} \chi^2 e^{-\frac{1}{2}Q} \left(e^{\frac{1}{2}Q} u \right)' \right). \tag{4.146}$$

As expected from ordinary 2d gravity C is conformally invariant.

Expressing the Poisson tensor in terms of the potentials $V(\phi)$ and $Z(\phi)$ of the original bosonic theory, and with $u(\phi)$ as in (4.144) we arrive at

$$\mathcal{P}^{ab} = \left(V + YZ - \frac{1}{2}\chi^2\left[\frac{VZ+V'}{2u} + \frac{\tilde{u}_0^2V^2}{2u^3}\right]\right)\epsilon^{ab}, \tag{4.147}$$

$$\mathcal{P}^{\alpha b} = \frac{Z}{4}X^a(\chi\gamma_a\gamma^b\gamma^3)^\alpha - \frac{i\tilde{u}_0V}{2u}(\chi\gamma^b)^\alpha, \tag{4.148}$$

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}_0X^c\gamma_c{}^{\alpha\beta} + \left(u + \frac{Z}{8}\chi^2\right)\gamma^{3\alpha\beta}. \tag{4.149}$$

As will be shown in Section 4.5.3 this provides a supersymmetrization for all the dilaton theories (1.3), because it covers all theories (1.2) with v linear in Y. Among these two explicit examples, namely SRG and the KV model, will be treated in more detail now.

Spherically Reduced Gravity (SRG), Nondiagonal Extension I

In contrast to the KV-model below, no obstructions are found when (4.126)–(4.128) with v given by (4.125) is used for SRG. For simplicity we take in (1.5) the case $d = 4$ and obtain $Q(\phi) = -\frac{1}{2}\ln(\phi)$, $W(\phi) = -2\lambda^2\sqrt{\phi}$ and

$$u = 2\tilde{u}_0\lambda\sqrt{\phi}, \qquad \varphi = \frac{1}{4}\ln(\phi), \tag{4.150}$$

where u_0 is a constant. Here already the bosonic theory is defined for $\phi > 0$ only. From (4.140)–(4.142) in the Poisson tensor of SRG

$$P^{ab} = \left(-\lambda^2 - \frac{Y}{2\phi} - \frac{3\lambda}{32\tilde{u}_0\phi^{3/2}}\chi^2\right)\epsilon^{ab}, \tag{4.151}$$

$$P^{\alpha b} = -\frac{1}{8\phi}X^c(\chi\gamma_c\gamma^b\gamma^3)^\alpha + \frac{i\lambda}{4\sqrt{\phi}}(\chi\gamma^b)^\alpha, \tag{4.152}$$

$$P^{\alpha\beta} = i\tilde{u}_0X^c\gamma_c{}^{\alpha\beta} + \left(2\tilde{u}_0\lambda\sqrt{\phi} - \frac{1}{16\phi}\chi^2\right)\gamma^{3\alpha\beta} \tag{4.153}$$

the singularity of the bosonic part simply carries over to the extension, without introducing any new restriction for $\phi > 0$.

The bosonic part of the Casimir function (4.146) is proportional to the ADM mass for SRG.

Katanaev-Volovich Model (KV), Nondiagonal Extension I

The bosonic potential (1.7) leads to $Q(\phi) = \alpha\phi$, thus $\varphi = -\frac{\alpha}{2}\phi$, and

$$W(\phi) = \int_{\phi_0}^{\phi} e^{\alpha\eta}\left(\frac{\beta}{2}\eta^2 - \Lambda\right)d\eta = e^{\alpha\eta}\left[\frac{\beta}{2}\left(\frac{2}{\alpha^3} - \frac{2\eta}{\alpha^2} + \frac{\eta^2}{\alpha}\right) - \frac{\Lambda}{\alpha}\right]\Bigg|_{\phi_0}^{\phi}. \tag{4.154}$$

With $u(\phi)$ calculated according to (4.144) the Poisson tensor is

$$\mathcal{P}^{ab} = \left(\frac{\beta}{2}\phi^2 - \Lambda + \alpha Y + \frac{1}{2}\chi^2 v_2\right)\epsilon^{ab}, \tag{4.155}$$

$$\mathcal{P}^{\alpha b} = \frac{\alpha}{4}X^a(\chi\gamma_a\gamma^b\gamma^3)_\alpha{}^\beta - \frac{i\tilde{u}_0}{2u}\left(\frac{\beta}{2}\phi^2 - \Lambda\right)(\chi\gamma^b)_\alpha{}^\beta, \tag{4.156}$$

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}_0 X^c \gamma_c{}^{\alpha\beta} + \left(u + \frac{\alpha}{8}\chi^2\right)\gamma^{3\alpha\beta}, \tag{4.157}$$

with

$$v_2 = -\frac{\alpha\left(\frac{\beta}{2}\phi^2 - \Lambda\right) + \beta\phi}{2u} - \frac{\tilde{u}_0^2\left(\frac{\beta}{2}\phi^2 - \Lambda\right)^2}{2u^3}. \tag{4.158}$$

For general parameters α, β, Λ from (4.144) restrictions upon the range of ϕ will in general emerge, if we do not allow singular and complex Poisson tensors. It may even happen that no allowed interval for ϕ exists. In fact, as we see from (4.158), in the present case, the ‘problem’ of complex-valued Poisson tensors comes together with the ‘singularity-problem’.

On the other hand, for $\beta \leq 0$ and $\Lambda \geq 0$, where at least one of these parameter does not vanish, the integrand in (4.154) becomes negative definite, leading to the restriction $\phi > \phi_0$ with singularities at $\phi = \phi_0$. If we further assume $\alpha > 0$ we can set $\phi_0 = -\infty$. In contrast to the torsionless R^2 model (see Table 4.1) the restriction for this particular case disappears and the fermionic potential becomes

$$u = \pm\tilde{u}_0\sqrt{-\frac{\beta}{\alpha^3}\left((1-\alpha\phi)^2 + 1\right) + \frac{2\Lambda}{\alpha}}. \tag{4.159}$$

4.4.6 General Prepotential Algebra

This algebra represents the immediate generalization of the torsionless one of Section 4.4.4, when (4.110) is taken for $v^{\alpha\beta}$, but now with u depending on both ϕ and Y. Here also $v = v(\phi, Y)$. Again we have $\Delta = 2Y\tilde{u}_0^2 - u^2$. By analogy to the step in Section 4.4.4 we again cancel the $f_{(t)}$ term (cf. (4.33)) by the choice

$$f^c = \frac{1}{2}(\nabla u)X^c. \tag{4.160}$$

This yields

$$F^c{}_\alpha{}^\beta = \frac{1}{2\Delta}\left(\tilde{u}_0^2 v + u\nabla u\right)X^a(\gamma_a\gamma^c\gamma^3)_\alpha{}^\beta + \frac{i\tilde{u}_0}{2\Delta}\left(uv + 2Y\nabla u\right)\gamma^c{}_\alpha{}^\beta, \tag{4.161}$$

$$v_2{}^{\alpha\beta} = \frac{1}{2\Delta}\left(\left(\tilde{u}_0^2 v + u\nabla u\right) + \dot{u}\left(uv + 2Y\nabla u\right)\right)\gamma^{3\alpha\beta}, \tag{4.162}$$

$$v_2 = \frac{uv}{2\Delta^2}\left(\tilde{u}_0^2 v + u\nabla u\right) + \frac{u\nabla u}{2\Delta^2}\left(uv + 2Y\nabla u\right) + \frac{1}{2\Delta}\left(uv' + 2Y\dot{v}\nabla u + 2Y\nabla^2 u\right). \tag{4.163}$$

The factors Δ^{-1}, Δ^{-2} indicate the appearance of action functional singularities, at values of the fields where Δ vanishes. Again we have to this point kept u independent of v. In this case, even when we relate v and u by imposing e.g.

$$\tilde{u}_0^2 v + u\nabla u = 0 \Leftrightarrow \nabla\Delta = 0 \Leftrightarrow c(\phi, Y) = \Delta, \tag{4.164}$$

in order to cancel the first terms in (4.161)–(4.163), a generic singularity obstruction is seen to persist (previous remarks on similar occasions should apply here, too, however).

4.4.7 Algebra with $u(\phi, Y)$ and $\tilde{u}(\phi, Y)$

For $v^{\alpha\beta}$ we retain (4.110), but now with both u and $\tilde{u}$ depending on ϕ and Y. Again the determinant

$$\Delta = 2Y\tilde{u}^2 - u^2 \tag{4.165}$$

will introduce singularities. If we want to cancel the $f_{(t)}$ term (cf. (4.33)) as we did in Section 4.4.4 and Section 4.4.6, we have to set here

$$f^c = \frac{1}{2\tilde{u}} \left(\tilde{u}\nabla u - u\nabla\tilde{u}\right) X^c. \tag{4.166}$$

This leads to

$$F^c = -\frac{1}{4}(\nabla \ln \Delta) X^a \gamma_a \gamma^c \gamma^3 + \frac{1}{2}(\nabla \ln \tilde{u}) X^c \gamma^3 + \frac{i\tilde{u}u}{2\Delta} \left[v + 2Y\left(\nabla \ln \frac{u}{\tilde{u}}\right)\right]\gamma^c. \tag{4.167}$$

Again we could try to fix u and $\tilde{u}$ suitably so as to cancel e.g. the first term in (4.167):

$$-\frac{1}{2}\nabla\Delta = \tilde{u}^2 v + u\nabla u - 2Y\tilde{u}\nabla\tilde{u} = 0. \tag{4.168}$$

But then the singularity obstruction resurfaces in (cf. (4.168))

$$v = \frac{\Delta'}{\dot{\Delta}} = \frac{-uu' + 2Y\tilde{u}\tilde{u}'}{\tilde{u}^2 - u\dot{u} + 2Y\tilde{u}\dot{\tilde{u}}}\ . \tag{4.169}$$

Eq. (4.167) becomes

$$F^c = \frac{1}{\dot{\Delta}}\left(\tilde{u}\tilde{u}' - \frac{u}{\tilde{u}}\left(\tilde{u}'\dot{u} - \dot{\tilde{u}}u'\right)\right) X^c\gamma^3 + \frac{i}{\dot{\Delta}}\left(\tilde{u}u' - 2Y\left(\tilde{u}'\dot{u} - \dot{\tilde{u}}u'\right)\right)\gamma^c. \tag{4.170}$$

The general formulae for the Poisson tensor are not very illuminating. Instead, we consider two special cases.

Spherically Reduced Gravity (SRG), Nondiagonal Extension II

For SRG also e.g. the alternative

$$v^{\mathrm{SRG}}(\phi, Y) = \frac{\Delta'}{\dot{\Delta}} \tag{4.171}$$

exists, where $\tilde{u}$ and u are given by

$$\tilde{u} = \frac{\tilde{u}_0}{\sqrt[4]{\phi}}, \qquad u = 2\tilde{u}_0 \lambda \sqrt[4]{\phi}\,, \tag{4.172}$$

and $\tilde{u}_0 = \text{const.}$ The Poisson tensor is

$$P^{ab} = \left(-\lambda^2 - \frac{Y}{2\phi} - \frac{3\lambda}{32\tilde{u}_0 \phi^{5/4}} \chi^2 \right) \epsilon^{ab}, \tag{4.173}$$

$$P^{\alpha b} = -\frac{1}{8\phi} X^b (\chi\gamma^3)^\alpha + \frac{i\lambda}{4\sqrt{\phi}} (\chi\gamma^b)^\alpha, \tag{4.174}$$

$$P^{\alpha\beta} = \frac{i\tilde{u}_0}{\sqrt[4]{\phi}} X^c {\gamma_c}^{\alpha\beta} + 2\tilde{u}_0 \lambda \sqrt[4]{\phi}\, \gamma^{3\alpha\beta}. \tag{4.175}$$

Regarding the absence of obstructions this solution is as acceptable as (and quite similar to) (4.151)–(4.153). Together with the diagonal extension implied by Section 4.4.1 and the nondegenerate chiral extension of Section 4.4.2, these four solutions for the extension of the physically motivated 2d gravity theory in themselves represent a counterexample to the eventual hope that the requirement for nonsingular, real extensions might yield a unique answer, especially also for a supersymmetric $N = 1$ extension of SRG.

Katanaev-Volovich Model (KV), Nondiagonal Extension II

Within the fermionic extension treated now also another alternative version of the KV case may be formulated. As for SRG in Section 4.4.7 we may identify the bosonic potential (1.7) with

$$v^{\mathrm{KV}}(\phi, Y) = \frac{\Delta'}{\dot{\Delta}}. \tag{4.176}$$

Then $\tilde{u}$ and u must be chosen as

$$\tilde{u} = \tilde{u}_0 e^{\frac{\alpha}{2}\phi}, \tag{4.177}$$

$$u = \pm\sqrt{-2\tilde{u}_0^2 W(\phi)}, \tag{4.178}$$

where $\tilde{u}_0 = \text{const}$ and $W(\phi)$ has been defined in (4.154). Instead of (4.155)–(4.157) we then obtain

$$\mathcal{P}^{ab} = \left(\frac{\beta}{2}\phi^2 - \Lambda + \alpha Y + \frac{1}{2}\chi^2 v_2\right)\epsilon^{ab}, \tag{4.179}$$

$$\mathcal{P}^{\alpha b} = \frac{\alpha}{4}X^b(\chi\gamma^3)_\alpha{}^\beta - \frac{i\tilde{u}}{2u}\left(\frac{\beta}{2}\phi^2 - \Lambda\right)(\chi\gamma^b)_\alpha{}^\beta, \tag{4.180}$$

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}X^c\gamma_c{}^{\alpha\beta} + u\gamma^{3\alpha\beta}. \tag{4.181}$$

with

$$v_2 = -\frac{\alpha\left(\frac{\beta}{2}\phi^2 - \Lambda\right) + \beta\phi}{2u} - \frac{\tilde{u}^2\left(\frac{\beta}{2}\phi^2 - \Lambda\right)^2}{2u^3}, \tag{4.182}$$

which, however, is beset with the same obstruction problems as the nondiagonal extension I.

4.5 Supergravity Actions

The algebras discussed in the last section have been selected in view of their application in specific gravitational actions.

4.5.1 First Order Formulation

With the notation introduced in Section 4.2.1, the identification (1.10), and after a partial integration, the action (4.12) takes the explicit form (remember $e_A = (e_a, \psi_\alpha)$)

$$L^{\text{gFOG}} = \int_{\mathcal{M}} \phi d\omega + X^a De_a + \chi^\alpha D\psi_\alpha + \frac{1}{2}\mathcal{P}^{AB}e_B e_A, \tag{4.183}$$

where the elements of the Poisson structure by expansion in Lorentz covariant components in the notation of Section 4.2.1 can be expressed explicitly as (cf. (4.26)–(4.33))

$$\begin{aligned}\frac{1}{2}\mathcal{P}^{AB}e_B e_A = &-\frac{1}{2}U(\psi\gamma^3\psi) - \frac{i}{2}\widetilde{U}X^a(\psi\gamma_a\psi) - \frac{i}{2}\widehat{U}X^a\epsilon_a{}^b(\psi\gamma_b\psi)\\ &+ (\chi F^a e_a\psi) + \frac{1}{2}V\epsilon^{ba}e_a e_b.\end{aligned} \tag{4.184}$$

Here $F^a \equiv F^a{}_\beta{}^\gamma$, the quantity of (4.30), provides the direct coupling of ψ and χ, and D is the Lorentz covariant exterior derivative,

$$DX^a = dX^a + X^b\omega\epsilon_b{}^a, \qquad D\chi^\alpha = d\chi^\alpha - \frac{1}{2}\chi^\beta\omega\gamma^3{}_\beta{}^\alpha. \tag{4.185}$$

Of course, at this point the Jacobi identity had not been used as yet to relate the arbitrary functions; hence the action functional (4.184) is not

invariant under a local supersymmetry. On the other hand, when the Jacobi identities restrict those arbitrary functions, the action (4.183) possesses the local symmetries (4.15), where the parameters $\epsilon_I = (l, \epsilon_a, \epsilon_\alpha)$ correspond to Lorentz symmetry, diffeomorphism invariance and, in addition, to supersymmetry, respectively. Nevertheless, already at this point we may list the explicit supersymmetry transformations with parameter $\epsilon_I = (0, 0, \epsilon_\alpha)$ for the scalar fields,

$$\delta\phi = \frac{1}{2}(\chi\gamma^3\epsilon), \tag{4.186}$$

$$\delta X^a = -(\chi F^a \epsilon), \tag{4.187}$$

$$\delta\chi^\alpha = U(\gamma^3\epsilon)^\alpha + i\widetilde{U}X^c(\gamma_c\epsilon)^\alpha + i\widehat{U}X^d\epsilon_d{}^c(\gamma_c\epsilon)^\alpha, \tag{4.188}$$

and also for the gauge fields,

$$\delta\omega = U'(\epsilon\gamma^3\psi) + i\widetilde{U}'X^b(\epsilon\gamma_b\psi) + i\widehat{U}'X^a\epsilon_a{}^b(\epsilon\gamma_b\psi) + (\chi\partial_\phi F^b\epsilon)e_b, \tag{4.189}$$

$$\begin{aligned} \delta e_a = {} & i\widetilde{U}(\epsilon\gamma_a\psi) + i\widehat{U}\epsilon_a{}^b(\epsilon\gamma_b\psi) + (\chi\partial_a F^b\epsilon)e_b \\ & + X_a\left[\dot{U}(\epsilon\gamma^3\psi) + i\dot{\widetilde{U}}X^b(\epsilon\gamma_b\psi) + i\dot{\widehat{U}}X^b\epsilon_b{}^c(\epsilon\gamma_c\psi)\right], \end{aligned} \tag{4.190}$$

$$\begin{aligned} \delta\psi_\alpha = {} & -D\epsilon_\alpha + (F^b\epsilon)_\alpha e_b \\ & + \chi_\alpha\left[u_2(\epsilon\gamma^3\psi) + i\tilde{u}_2X^b(\epsilon\gamma_b\psi) + i\hat{u}_2X^a\epsilon_a{}^b(\epsilon\gamma_b\psi)\right], \end{aligned} \tag{4.191}$$

with the understanding that they represent symmetries of the action (4.183) only after the relations between the still arbitrary functions for some specific algebra are implied. The only transformation independent of those functions is (4.186).

4.5.2 Elimination of the Auxiliary Fields X^I

We can eliminate the fields X^I by a Legendre transformation. To sketch the procedure, we rewrite the action (4.12) in a suggestive form as Hamiltonian action principle ($d^2x = dx^1 \wedge dx^0$)

$$L = \int_{\mathcal{M}} d^2x \left(X^I\dot{\mathcal{A}}_I - \mathcal{H}(X, A)\right), \tag{4.192}$$

where X^I should be viewed as the 'momenta' conjugate to the 'velocities' $\dot{\mathcal{A}}_I = \partial_0 A_{1I} - \partial_1 A_{0I}$ and A_{mI} as the 'coordinates'. Velocities $\dot{\mathcal{A}}_I$ and the 'Hamiltonian' $\mathcal{H}(X, A) = \mathcal{P}^{JK}A_{0K}A_{1J}$ are densities in the present definition. The second PSM field equation (4.14), in the form obtained when varying X^I in (4.192), acts as a Legendre transformation of $\mathcal{H}(X, A)$ with respect to the variables X^I,

$$\dot{\mathcal{A}}_I = \frac{\vec{\partial}\mathcal{H}(X, A)}{\partial X^I}, \tag{4.193}$$

also justifying the interpretation of $\dot{\mathcal{A}}_I$ as conjugate to X^I. When (4.193) can be solved for all X^I, we get $X^I = X^I(\dot{A}, A)$. Otherwise, not all of the X^I can be eliminated and additional constraints $\Phi(\dot{\mathcal{A}}, A) = 0$ emerge. In the latter situation the constraints may be used to eliminate some of the gauge fields A_I in favour of others. When all X^I can be eliminated the Legendre transformed density

$$\mathcal{F}(\dot{\mathcal{A}}, A) = X^I(\dot{\mathcal{A}}, A)\dot{\mathcal{A}}_I - \mathcal{H}(X(\dot{\mathcal{A}}, A), A) \tag{4.194}$$

follows, as well as the second order Lagrangian action principle

$$L = \int_{\mathcal{M}} d^2x\, \mathcal{F}(\dot{\mathcal{A}}, A), \tag{4.195}$$

where the coordinates A_{mI} must be varied independently.

The formalism already presented applies to any (graded) PSM. If there is an additional volume form ϵ on the base manifold $\mathcal{M}$ it may be desirable to work with functions instead of densities. This is also possible if the volume is dynamical as in gravity theories, $\epsilon = \epsilon(A)$, because a redefinition of the velocities $\dot{\mathcal{A}}_I$ containing coordinates A_{mI} but not momenta X^I is always possible, as long as we can interpret the field equations from varying X^I as Legendre transformation. In particular we use $\dot{A}_I = \star dA_I = \epsilon^{mn}\partial_n A_{mI}$ as velocities and $H(X, A) = \star\left(-\frac{1}{2}\mathcal{P}^{JK}A_K A_J\right) = \frac{1}{2}P^{JK}\epsilon^{mn}A_{nK}A_{mJ}$ as Hamiltonian function, yielding

$$L = \int_{\mathcal{M}} \epsilon\left(X^I \dot{A}_I - H(X, A)\right). \tag{4.196}$$

Variation of X^I leads to

$$\dot{A}_I = \frac{\vec{\partial} H(X, A)}{\partial X^I}. \tag{4.197}$$

Solving this equation for $X^I = X^I(\dot{A}, A)$ the Legendre transformed function $F(\dot{A}, A) = X^I(\dot{A}, A)\dot{A}_I - H(X(\dot{A}, A), A)$ constitutes the action

$$L = \int_{\mathcal{M}} \epsilon F(\dot{A}, A). \tag{4.198}$$

If the Poisson tensor is linear in the coordinates $\mathcal{P}^{JK} = X^I f_I{}^{JK}$, where $f_I{}^{JK}$ are structure constants, (4.197) cannot be used to solve for X^I, instead the constraint $\dot{A}_I - \frac{1}{2}f_I{}^{JK}\epsilon^{mn}A_{nK}A_{mJ} = 0$ appears, implying that the field strength of ordinary gauge theory is zero. For a nonlinear Poisson tensor we always have the freedom to move X^I-independent terms from the r.h.s. to the l.h.s. of (4.197), thus using this particular type of covariant derivatives as velocities conjugate to the momenta X^I in the Legendre transformation.

This redefinition can already be done in the initial action (4.196) leading to a redefinition of the Hamiltonian $H(X, A)$.

In order to bring 2d gravity theories into the form (4.196), but with covariant derivatives, it is desirable to split off ϕ-components of the Poisson tensor and to define the 'velocities' (cf. (A.11) and (A.13) in Appendix A) as

$$\rho := \star d\omega = \epsilon^{mn}(\partial_n \omega_m), \tag{4.199}$$

$$\tau_a := \star De_a = \epsilon^{mn}(\partial_n e_{ma}) - \omega_a, \tag{4.200}$$

$$\sigma_\alpha := \star D\psi_\alpha = \epsilon^{mn}\left(\partial_n \psi_{m\alpha} + \frac{1}{2}\omega_n(\gamma^3\psi_m)_\alpha\right). \tag{4.201}$$

Here $\rho = R/2$ is proportional to the Ricci scalar; τ_a and σ_α are the torsion vector and the spinor built from the derivative of the Rarita-Schwinger field, respectively. As a consequence the Lorentz connection ω_m is absent in the Hamiltonian,

$$V(\phi, X^A; e_{mA}) = \frac{1}{2}\mathcal{P}^{BC}\epsilon^{mn}e_{nC}e_{mB} = \frac{1}{e}\mathcal{P}^{BC}e_{0C}e_{1B}, \tag{4.202}$$

of the supergravity action

$$L = \int_{\mathcal{M}} \epsilon\left(\phi\rho + X^a\tau_a + \chi^\alpha\sigma_\alpha - V(\phi, X^A; e_{mA})\right). \tag{4.203}$$

4.5.3 Superdilaton Theory

As remarked already in Section 1, first order formulations of (bosonic) 2d gravity (and hence PSMs) allow for an—at least on the classical level—globally equivalent description of general dilaton theories (1.3). Here we show that this statement remains valid also in the case of additional supersymmetric partners (i.e. for gPSMs). We simply have to eliminate the Lorentz connection ω_a and the auxiliary field X^a. Of course, also the validity of an algebraic elimination procedure in the most general case should (and can) be checked by verifying that the correct e.o.m.s also follow from the final action (4.212) or (4.213). (Alternatively to the procedure applied below one may also proceed as in [79], performing two 'Gaussian integrals' to eliminate X^a and τ^a from the action). In fact, in the present section we will allow also for Poisson structures characterized by a bosonic potential v not necessarily linear in $Y \equiv \frac{1}{2}X^aX_a$ as in (1.4).

Variation of X^a in (4.183) yields the torsion equation

$$\tau_a = \frac{1}{2}(\partial_a\mathcal{P}^{AB})\epsilon^{mn}e_{nB}e_{mA}. \tag{4.204}$$

From (4.204) using $\tilde{\omega}_a := \star de_a = \epsilon^{mn}(\partial_n e_{ma})$ and $\tau_a = \tilde{\omega}_a - \omega_a$ (cf. (4.200))[5] we get

$$\omega_a = \tilde{\omega}_a - \frac{1}{2}(\partial_a \mathcal{P}^{AB})\epsilon^{mn} e_{nB} e_{mA}. \tag{4.205}$$

Using (4.205) to eliminate ω_a the separate terms of (4.183) read

$$\phi d\omega = \phi d\tilde{\omega} + \epsilon\epsilon^{an}(\partial_n \phi)\tau_a + \text{total div.}, \tag{4.206}$$
$$X^a De_a = \epsilon X^a \tau_a, \tag{4.207}$$
$$\chi^\alpha D\psi_\alpha = \chi^\alpha \tilde{D}\psi_\alpha + \frac{1}{2}\epsilon\epsilon^{an}(\chi\gamma^3\psi_n)\tau_a, \tag{4.208}$$

where $\tau_a = \tau_a(X^I, e_{mA})$ is determined by (4.204). The action, discarding the boundary term in (4.206), becomes

$$L = \int_{\mathcal{M}} d^2x e \Bigg[\phi\tilde{\rho} + \chi^\alpha \tilde{\sigma}_\alpha - \frac{1}{2}\mathcal{P}^{AB}\epsilon^{mn} e_{nB} e_{mA} \\ + \left(X^a + \epsilon^{ar}\partial_r \phi + \frac{1}{2}\epsilon^{ar}(\chi\gamma^3\psi_r) \right) \frac{1}{2}(\partial_a \mathcal{P}^{AB})\epsilon^{mn} e_{nB} e_{mA} \Bigg]. \tag{4.209}$$

Here $\tilde{\rho}$ and $\tilde{\sigma}$ are defined in analogy to (4.199) and (4.201), but calculated with $\tilde{\omega}_a$ instead of ω_a.

To eliminate X^a we vary once more with respect to δX^b:

$$\left[X^a + \epsilon^{an}(\partial_n \phi) + \frac{1}{2}\epsilon^{an}(\chi\gamma^3\psi_n) \right] (\partial_b \partial_a \mathcal{P}^{AB})\epsilon^{mn} e_{nB} e_{mA} = 0. \tag{4.210}$$

For $(\partial_b \partial_a \mathcal{P}^{AB})\epsilon^{mn} e_{nB} e_{mA} \neq 0$ this leads to the (again *algebraic*) equation

$$X^a = -\epsilon^{an} \left[(\partial_n \phi) + \frac{1}{2}(\chi\gamma^3\psi_n) \right] \tag{4.211}$$

for X^a. It determines X^a in a way which does not depend on the specific gPSM, because (4.211) is nothing else than the e.o.m. for $\delta\omega$ in (4.183).

We thus arrive at the superdilaton action for an arbitrary gPSM

$$L = \int_{\mathcal{M}} \phi d\tilde{\omega} + \chi^\alpha \tilde{D}\psi_\alpha + \frac{1}{2}\mathcal{P}^{AB}\bigg|_{X^a} e_B e_A, \tag{4.212}$$

where $|_{X^a}$ means that X^a has to be replaced by (4.211). The action (4.212) expressed in component fields

$$L = \int_{\mathcal{M}} d^2x e \left[\phi \frac{\tilde{R}}{2} + \chi^\alpha \tilde{\sigma}_\alpha - \frac{1}{2}\mathcal{P}^{AB}\bigg|_{X^a} \epsilon^{mn} e_{nB} e_{mA} \right] \tag{4.213}$$

[5] For specific supergravities it may be useful to base this separation upon a different SUSY-covariant Lorentz connection $\breve{\omega}_a$ (cf. Section 4.6.4 and 4.6.5).

explicitly shows the fermionic generalization of the bosonic dilaton theory (1.3) for any gPSM. Due to the quadratic term $X^2 = X^a X_a$ in $\mathcal{P}^{AB}$ the first term in (4.211) provides the usual kinetic term of the ϕ-field in (1.3) if we take the special case (1.4),

$$L = L^{\mathrm{dil}} + L^{\mathrm{f}} \tag{4.214}$$

$$L^{\mathrm{f}} = \int_{\mathcal{M}} d^2x e \Big[\chi^\alpha \tilde{\sigma}_\alpha - \frac{Z}{2}(\partial^n \phi)(\chi \gamma^3 \psi_n) + \frac{Z}{16}\chi^2(\psi^n \psi_n) + \frac{1}{2}\chi^2 v_2 \Big|_{X^a} + (\chi F^a \Big|_{X^a} \epsilon_a{}^m \psi_m) - \frac{1}{2}\mathcal{P}^{\alpha\beta}\Big|_{X^a} \epsilon^{mn} \psi_{n\beta}\psi_{m\alpha} \Big]. \tag{4.215}$$

However, (4.213) even allows an arbitrary dependence on X^2 and a corresponding dependence on higher powers of $(\partial^n \phi)(\partial_n \phi)$. For the special case where $\mathcal{P}^{AB}$ is linear in X^a (4.209) shows that X^a drops out of that action without further elimination. However, the final results (4.212) and (4.213) are the same.

4.6 Actions for Particular Models

Whereas in Section 4.4 a broad range of solutions of graded Poisson algebras has been constructed, we now discuss the related actions and their (eventual) relation to a supersymmetrization of (1.1) or (1.3). It will be found that, in contrast to the transition from (1.2) to (1.3) the form (1.1) of the supersymmetric action requires that the different functions in the gPSM solution obey certain conditions which are not always fulfilled.

For example, in order to obtain the supersymmetrization of (1.1), ϕ and X^a should be eliminated by a Legendre transformation. This is possible only if the Hessian determinant of $v(\phi, Y)$ with respect to $X^i = (\phi, X^a)$ is regular,

$$\det\left(\frac{\partial^2 v}{\partial X^i \partial X^j}\right) \neq 0. \tag{4.216}$$

Even in that case the generic situation will be that no closed expression of the form (1.1) can be given.

In the following subsections for each algebra of Section 4.4 the corresponding FOG action (4.183), the related supersymmetry and examples of the superdilaton version (4.213) will be presented. In the formulae for the local supersymmetry we always drop the (common) transformation law for $\delta\phi$ (4.186).

4.6.1 Block Diagonal Supergravity

The action functional can be read off from (4.183) and (4.184) for the Poisson tensor of Section 4.4.1 (cf. (4.94) for U and (4.92) together with (4.29) to

determine V). It reads

$$L^{\mathrm{BDS}} = \int_{\mathcal{M}} \phi d\omega + X^a De_a + \chi^\alpha D\psi_\alpha - \frac{1}{2}U(\psi\gamma^3\psi) + \frac{1}{2}V\epsilon^{ba}e_a e_b, \quad (4.217)$$

and according to (4.186)–(4.191) possesses the local supersymmetry

$$\delta X^a = 0, \qquad \delta\chi^\alpha = U(\gamma^3\epsilon)^\alpha, \quad (4.218)$$

$$\delta\omega = U'(\epsilon\gamma^3\psi), \qquad \delta e_a = X_a\dot{U}(\epsilon\gamma^3\psi), \qquad \delta\psi_\alpha = -D\epsilon_\alpha + \chi_\alpha u_2(\epsilon\gamma^3\psi). \quad (4.219)$$

This transformation leads to a translation of χ^α on the hypersurface $C =$ const if $u \neq 0$. Comparing with the usual supergravity type symmetry (1.15) we observe that the first term in $\delta\psi_\alpha$ has the required form (1.15), but the variation δe_a is quite different.

The fermionic extension (4.93) of the Casimir function (4.21) for this class of theories implies an absolute conservation law $C = c_0 =$ const.

Whether the supersymmetric extension of the action of type (1.1) can be reached depends on the particular choice of the bosonic potential v. An example where the elimination of all target space coordinates ϕ, X^a and χ^α is feasible and actually quite simple is R^2-supergravity with $v = -\frac{\alpha}{2}\phi^2$ and $U = u_0 =$ const.[6] The result in this case is (cf. Section 4.5.2 and especially (4.203))

$$L^{R^2} = \int_{\mathcal{M}} d^2xe \left[\frac{1}{8\alpha}\tilde{R}^2 - \frac{2u_0}{\tilde{R}}\tilde{\sigma}^\alpha\tilde{\sigma}_\alpha + \frac{u_0}{2}\epsilon^{nm}(\psi_m\gamma^3\psi_n)\right]. \quad (4.220)$$

Here the tilde in $\tilde{R}$ and $\tilde{\sigma}^\alpha$ indicates that the torsion-free connection $\tilde{\omega}_a = \epsilon^{mn}\partial_n e_{ma}$ is used to calculate the field strengths. In supergravity it is not convenient to eliminate the field ϕ. Instead it should be viewed as the 'auxiliary field' of supergravity and therefore remain in the action. Thus eliminating only X^a and χ^α yields

$$L^{R^2} = \int_{\mathcal{M}} d^2xe \left[\phi\frac{\tilde{R}}{2} - \frac{u_0}{\alpha\phi}\tilde{\sigma}^\alpha\tilde{\sigma}_\alpha - \frac{\alpha}{2}\phi^2 + \frac{u_0}{2}\epsilon^{nm}(\psi_m\gamma^3\psi_n)\right]. \quad (4.221)$$

Also for SRG in d-dimensions (1.5) such an elimination is possible if $d \neq 4$, $d < \infty$. However, interestingly enough, the Hessian determinant vanishes just for the physically most relevant dimension four (SRG) and for the DBH (1.9), preventing in this case a transition to the form (1.1).

The formula for the equivalent superdilaton theories (4.213) is presented for the restriction (1.4) to quadratic torsion only, in order to have a direct comparison with (1.3). The choice $U = u_0 =$ const yields

$$L^{\mathrm{BDA}} = L^{\mathrm{dil}} + L^{\mathrm{f}} \quad (4.222)$$

[6] Clearly then no genuine supersymmetry is implied by (4.219). But we use this example as an illustration for a complete elimination of different combinations of ϕ, X^a and χ^α.

with the fermionic extension

$$L^{\mathrm{f}} = \int_{\mathcal{M}} d^2xe\Big[\chi^\alpha\tilde{\sigma}_\alpha - \frac{Z}{2}(\partial^n\phi)(\chi\gamma^3\psi_n) + \frac{Z}{16}\chi^2(\psi^n\psi_n) - \frac{1}{4u_0}\chi^2\left(V' - \frac{Z'}{2}(\partial^n\phi)(\partial_n\phi)\right) + \frac{u_0}{2}\epsilon^{mn}(\psi_n\gamma^3\psi_m)\Big]. \quad (4.223)$$

It should be noted that this model represents a superdilaton theory for arbitrary functions $V(\phi)$ and $Z(\phi)$ in (1.3).

4.6.2 Parity Violating Supergravity

The action corresponding to the algebra of Section 4.4.2 inserted into (4.183) and (4.184) becomes

$$L^{\mathrm{PVS}} = \int_{\mathcal{M}} \phi d\omega + X^a De_a + \chi^\alpha D\psi_\alpha + \epsilon\left(v - \frac{1}{4u_0}\chi^2 v'\right) - \frac{i\tilde{u}_0 v}{2u_0}(\chi\gamma^a P_\pm e_a\psi) - \frac{i\tilde{u}_0}{2}X^a(\psi\gamma_a P_\pm\psi) - \frac{u_0}{2}(\psi\gamma^3\psi), \quad (4.224)$$

with local supersymmetry

$$\delta X^a = \frac{i\tilde{u}_0 v}{2u_0}(\chi\gamma^a P_\pm\epsilon), \qquad \delta\chi^\alpha = i\tilde{u}_0 X^b(\gamma_b P_\pm\epsilon)^\alpha + u_0(\gamma^3\epsilon)^\alpha, \quad (4.225)$$

as well as

$$\delta\omega = -\frac{i\tilde{u}_0 v'}{2u_0}(\chi\gamma^b P_\pm\epsilon)e_b, \quad (4.226)$$

$$\delta e_a = i\tilde{u}_0(\epsilon\gamma_a P_\pm\psi) - \frac{i\tilde{u}_0}{2u_0}X_a\dot{v}(\chi\gamma^b P_\pm\epsilon)e_b, \quad (4.227)$$

$$\delta\psi_\alpha = -D\epsilon_\alpha - \frac{i\tilde{u}_0 v}{2u_0}(\gamma^b P_\pm\epsilon)_\alpha e_b, \quad (4.228)$$

and the absolute conservation law (4.103). Here, in contrast to the model of Section 4.6.1, the transformation law of e_a essentially has the ‘canonical’ form (1.15).

As seen from the action (4.224) and the symmetry transformations the two chiralities are treated differently, but we do not have the case of a genuine chiral supergravity (cf. Section 4.6.6 below).

4.6.3 Deformed Rigid Supersymmetry

In this case with the algebra (4.107)–(4.109) we obtain

$$L^{\mathrm{DRS}} = \int_{\mathcal{M}} \phi d\omega + X^a De_a + \chi^\alpha D\psi_\alpha + \epsilon v + \frac{v}{4Y}X^a(\chi\gamma^3\gamma_a\gamma^b e_b\psi) - \frac{i\tilde{u}_0}{2}X^a(\psi\gamma_a\psi) - \frac{v}{16Y}\chi^2(\psi\gamma^3\psi) \quad (4.229)$$

with local supersymmetry (4.186),

$$\delta X^a = -\frac{v}{4Y} X^b (\chi \gamma_b \gamma^a \gamma^3 \epsilon), \qquad \delta \chi^\alpha = i\tilde{u}_0 X^b (\gamma_b \epsilon)^\alpha + \frac{1}{2}\chi^2 \frac{v}{4Y} (\gamma^3 \epsilon)^\alpha, \tag{4.230}$$

and

$$\delta\omega = \Big(\frac{v}{4Y}\Big)' \Big[X^c (\chi \gamma_c \gamma^b \gamma^3 \epsilon) e_b + \frac{1}{2}\chi^2 (\epsilon \gamma^3 \psi) \Big], \tag{4.231}$$

$$\begin{aligned} \delta e_a &= i\tilde{u}_0 (\epsilon \gamma_a \psi) + \frac{v}{4Y} (\chi \gamma_a \gamma^b \gamma^3 \epsilon) e_b \\ &\quad + X_a \Big(\frac{v}{4Y}\Big)^{\cdot} \Big[X^c (\chi \gamma_c \gamma^b \gamma^3 \epsilon) e_b + \frac{1}{2}\chi^2 (\epsilon \gamma^3 \psi) \Big], \end{aligned} \tag{4.232}$$

$$\delta\psi_\alpha = -D\epsilon_\alpha + \frac{v}{4Y} \Big[X^c (\gamma_c \gamma^b \gamma^3 \epsilon)_\alpha e_b + \chi_\alpha (\epsilon \gamma^3 \psi) \Big]. \tag{4.233}$$

Clearly, this model exhibits a 'genuine' supergravity symmetry (1.15). As pointed out already in Section 4.4.3 the bosonic potential $v(\phi, Y)$ is not restricted in any way by the super-extension. However, a new singularity of the action functional occurs at $Y = 0$. The corresponding superdilaton theory can be derived along the lines of Section 4.5.3.

4.6.4 Dilaton Prepotential Supergravities

In its FOG version the action from (4.183) with (4.121)–(4.123) reads

$$\begin{aligned} L^{\mathrm{DPA}} = \int_{\mathcal{M}} & \phi d\omega + X^a D e_a + \chi^\alpha D\psi_\alpha - \frac{1}{2\tilde{u}_0^2} \epsilon \left((u^2)' - \frac{1}{2}\chi^2 u'' \right) \\ & + \frac{iu'}{2\tilde{u}_0} (\chi \gamma^a e_a \psi) - \frac{i\tilde{u}_0}{2} X^a (\psi \gamma_a \psi) - \frac{u}{2} (\psi \gamma^3 \psi), \end{aligned} \tag{4.234}$$

where $\tilde{u}_0 = \text{const}$ and the 'prepotential' u depends on ϕ only. The corresponding supersymmetry becomes (4.186),

$$\delta X^a = -\frac{iu'}{2\tilde{u}_0} (\chi \gamma^a \epsilon), \qquad \delta \chi^\alpha = i\tilde{u}_0 X^b (\gamma_b \epsilon)^\alpha + u (\gamma^3 \epsilon)^\alpha, \tag{4.235}$$

and further

$$\delta\omega = u' (\epsilon \gamma^3 \psi) + \frac{iu''}{2\tilde{u}_0} (\chi \gamma^b \epsilon) e_b, \tag{4.236}$$

$$\delta e_a = i\tilde{u}_0 (\epsilon \gamma_a \psi), \tag{4.237}$$

$$\delta\psi_\alpha = -D\epsilon_\alpha + \frac{iu'}{2\tilde{u}_0} (\gamma^b \epsilon)_\alpha e_b. \tag{4.238}$$

Here we have the special situation of an action linear in X^a, described at the end of Section 4.5.3. Variation of X^a in (4.234) leads to the constraint

$De_a - \frac{i\tilde{u}_0}{2}(\psi\gamma_a\psi) = 0$. It can be used to eliminate the Lorentz connection, i.e. $\omega_a = \breve{\omega}_a$, where we introduced the SUSY-covariant connection

$$\breve{\omega}_a := \epsilon^{mn}(\partial_n e_{ma}) + \frac{i\tilde{u}_0}{2}\epsilon^{mn}(\psi_n\gamma_a\psi_m). \tag{4.239}$$

The action reads

$$L^{\mathrm{DPA}} = \int_{\mathcal{M}} d^2x e\Big[\phi\frac{\breve{R}}{2} + \chi^\alpha\breve{\sigma}_\alpha - \frac{1}{2\tilde{u}_0^2}\Big((u^2)' - \frac{1}{2}\chi^2 u''\Big) + \frac{iu'}{2\tilde{u}_0}(\chi\gamma^m\epsilon_m{}^n\psi_n) + \frac{u}{2}\epsilon^{mn}(\psi_n\gamma^3\psi_m)\Big], \tag{4.240}$$

where $\breve{R}$ and $\breve{\sigma}$ indicate that these covariant quantities are built with the spinor dependent Lorentz connection (4.239).

The present model is precisely the one studied in [129–132]. In Section 4.7 we give the explicit solution of the PSM field equations for this model. The R^2 model and the model of Howe will be treated in a little more detail now.

R^2 Model

The supersymmetric extension of R^2-gravity, where $v = -\frac{\alpha}{2}\phi^2$, is obtained with the general solution $u = \pm\tilde{u}_0\sqrt{\frac{\alpha}{3}(\phi^3 - \phi_0^3)}$ (cf. (4.124)).

In order to simplify the analysis we choose $u = \tilde{u}_0\sqrt{\frac{\alpha}{3}\phi^3}$. The parameter α can have both signs, implying the restriction on the range of the dilaton field such that $\alpha\phi > 0$. Thus the superdilaton action (4.240) becomes

$$L^{R^2} = \int_{\mathcal{M}} d^2x e\Big[\phi\frac{\breve{R}}{2} + \chi^\alpha\breve{\sigma}_\alpha - \frac{\alpha}{2}\phi^2 + \frac{1}{16\tilde{u}_0}\chi^2\sqrt{\frac{3\alpha}{\phi}} + \frac{3i\alpha}{4}\sqrt{\frac{\phi}{3\alpha}}\,(\chi\gamma^m\epsilon_m{}^n\psi_n) + \frac{\tilde{u}_0}{2}\sqrt{\frac{\alpha}{3}\phi^3}\,\epsilon^{mn}(\psi_n\gamma^3\psi_m)\Big]. \tag{4.241}$$

The equation obtained when varying χ^α yields

$$\chi^\alpha = -8\tilde{u}_0\sqrt{\frac{\phi}{3\alpha}}\,\breve{\sigma}^\alpha - 2i\tilde{u}_0(\psi^n\epsilon_n{}^m\gamma_m)^\alpha. \tag{4.242}$$

Eliminating the χ^α field gives

$$L^{R^2} = \int_{\mathcal{M}} d^2x e\Big[\phi\frac{\breve{R}}{2} - 4\tilde{u}_0\sqrt{\frac{\phi}{3\alpha}}\,\breve{\sigma}^\alpha\breve{\sigma}_\alpha - 2i\tilde{u}_0\phi(\breve{\sigma}\gamma^m\epsilon_m{}^n\psi_n) - \frac{\alpha}{2}\phi^2 + \frac{\tilde{u}_0}{4}\sqrt{\frac{\alpha}{3}\phi^3}\,\big(3(\psi^n\psi_n) - \epsilon^{mn}(\psi_n\gamma^3\psi_m)\big)\Big]. \tag{4.243}$$

Further elimination of ϕ requires the solution of a cubic equation for $\sqrt{\phi}$ with a complicated explicit solution, leading to an equally complicated supergravity generalization of the formulation (1.1) of this model.

Model of Howe

The supergravity model of Howe [46], originally derived in terms of superfields, is just a special case of our generic model in the graded PSM approach. Using for the various independent potentials the particular values

$$\tilde{u}_0 = -2, \qquad u = -\phi^2 \tag{4.244}$$

we obtain $\Delta = 8Y - \phi^4$ and for the other nonzero potentials (cf. (4.117), (4.118))

$$v = -\frac{1}{2}\phi^3, \qquad v_2 = -\frac{1}{4}, \qquad f_{(s)} = \frac{1}{2}\phi. \tag{4.245}$$

The Lagrange density for this model in the formulation (1.2) is a special case of (4.234):

$$\begin{aligned} L^{\text{Howe}} = \int_{\mathcal{M}} & \phi d\omega + X^a De_a + \chi^\alpha D\psi_\alpha \\ & + \frac{1}{2}\phi^2(\psi\gamma^3\psi) + iX^a(\psi\gamma_a\psi) + \frac{i}{2}\phi(\chi\gamma^a e_a\psi) - \frac{1}{2}\epsilon\left(\phi^3 + \frac{1}{4}\chi^2\right). \end{aligned} \tag{4.246}$$

The local supersymmetry transformations from (4.235)–(4.238) are

$$\delta X^a = -\frac{i}{2}\phi(\chi\gamma^a\epsilon), \qquad \delta\chi^\alpha = -\phi^2(\gamma^3\epsilon)^\alpha - 2iX^b(\gamma_b\epsilon)^\alpha, \tag{4.247}$$

and

$$\delta\omega = -2\phi(\epsilon\gamma^3\psi) + \frac{i}{2}(\chi\gamma^b\epsilon)e_b, \tag{4.248}$$

$$\delta e_a = -2i(\epsilon\gamma_a\psi), \tag{4.249}$$

$$\delta\psi_\alpha = -D\epsilon_\alpha + \frac{i}{2}\phi(\gamma^b\epsilon)_\alpha e_b. \tag{4.250}$$

Starting from the dilaton action (4.240) with (4.244) and (4.245), the remaining difference to the formulation of Howe is the appearance of the fermionic coordinate χ^α. Due to the quadratic term of χ^α in (4.246) we can use its own algebraic field equations to eliminate it. Applying the Hodge-dual yields

$$\chi_\alpha = 4\breve{\sigma}_\alpha + 2i\phi(\gamma^n\epsilon_n{}^m\psi_m)_\alpha. \tag{4.251}$$

Inserting this into the Lagrange density (4.246) and into the supersymmetry transformations (4.186) as well as into (4.247)–(4.250) and identifying ϕ with the scalar, usually interpreted as auxiliary field A, $\phi \equiv A$, reveals precisely the supergravity model of Howe. That model, in a notation almost identical to the one used here, is also contained in [141], where a superfield approach was used.

4.6.5 Supergravities with Quadratic Bosonic Torsion

The algebra (4.147)–(4.149) in (4.183) leads to

$$L^{\mathrm{QBT}} = \int_{\mathcal{M}} \phi d\omega + X^a De_a + \chi^\alpha D\psi_\alpha + \epsilon\left(V + \frac{1}{2}X^a X_a Z + \frac{1}{2}\chi^2 v_2\right) + \frac{Z}{4}X^a(\chi\gamma^3\gamma_a\gamma^b e_b\psi) - \frac{i\tilde{u}_0 V}{2u}(\chi\gamma^a e_a\psi) - \frac{i\tilde{u}_0}{2}X^a(\psi\gamma_a\psi) - \frac{1}{2}\left(u + \frac{Z}{8}\chi^2\right)(\psi\gamma^3\psi), \quad (4.252)$$

with $u(\phi)$ determined from $V(\phi)$ and $Z(\phi)$ according to (4.144) and

$$v_2 = -\frac{1}{2u}\left(VZ + V' + \frac{\tilde{u}_0^2 V^2}{u^2}\right). \quad (4.253)$$

The special interest in models of this type derives from the fact that because of their equivalence to the dilaton theories with dynamical dilaton field (cf. Section 4.5.3) they cover a large class of physically interesting models. Also, as shown in Section 4.4.5 these models are connected by a simple conformal transformation to theories without torsion, discussed in Section 4.6.4.

Regarding the action (4.252) it should be kept in mind that calculating the prepotential $u(\phi)$ we discovered the condition $W(\phi) < 0$ (cf. (4.144)). This excludes certain bosonic theories from supersymmetrization with real actions, and it may lead to restrictions on ϕ, but there is even more information contained in this inequality: It leads also to a restriction on Y. Indeed, taking into account (4.23) we find

$$Y > c(\phi, Y)e^{-Q(\phi)}. \quad (4.254)$$

The local supersymmetry transformations of the action (4.252) become (4.186),

$$\delta X^a = -\frac{Z}{4}X^b(\chi\gamma_b\gamma^a\gamma^3\epsilon) + \frac{i\tilde{u}_0 V}{2u}(\chi\gamma^a\epsilon), \quad (4.255)$$

$$\delta\chi^\alpha = i\tilde{u}_0 X^b(\gamma_b\epsilon)^\alpha + \left(u + \frac{Z}{8}\chi^2\right)(\gamma^3\epsilon)^\alpha, \quad (4.256)$$

and

$$\delta\omega = \left(-\frac{\tilde{u}_0^2 V}{u} - \frac{Zu}{2} + \frac{Z'}{8}\chi^2\right)(\epsilon\gamma^3\psi) + \frac{Z'}{4}X^a(\chi\gamma^3\gamma_a\gamma^b\epsilon)e_b - \frac{\tilde{u}_0}{2u}\left(V' + \frac{VZ}{2} + \frac{\tilde{u}_0^2 V^2}{u^2}\right)(\chi\gamma^b\epsilon)e_b, \quad (4.257)$$

$$\delta e_a = i\tilde{u}_0(\epsilon\gamma_a\psi) + \frac{Z}{4}(\chi\gamma^3\gamma_a\gamma^b\epsilon)e_b, \quad (4.258)$$

$$\delta\psi_\alpha = -D\epsilon_\alpha + \frac{Z}{4}X^a(\gamma^3\gamma_a\gamma^b\epsilon)_\alpha e_b - \frac{i\tilde{u}_0 V}{2u}(\gamma^b\epsilon)_\alpha e_b + \frac{Z}{4}\chi_\alpha(\epsilon\gamma^3\psi). \quad (4.259)$$

Finally, we take a closer look at the torsion condition. Variation of X^a in (4.252) yields

$$De_a - \frac{i\tilde{u}_0}{2}(\psi\gamma_a\psi) + \frac{Z}{4}(\chi\gamma^3\gamma_a\gamma^b e_b\psi) + \epsilon X_a Z = 0. \tag{4.260}$$

For $Z \neq 0$ this can be used to eliminate X^a directly, as described in Section 4.5.2 for a generic PSM. The general procedure to eliminate instead ω_a by this equation was outlined in Section 4.5.3. There, covariant derivatives were expressed in terms of $\tilde{\omega}_a$. For supergravity theories it is desirable to use SUSY-covariant derivatives instead. The standard covariant spinor dependent Lorentz connection $\breve{\omega}_a$ was given in (4.239). However, that quantity does not retain its SUSY-covariance if torsion is dynamical. Eq. (4.260) provides such a quantity. Taking the Hodge dual, using (4.200) we find

$$\omega_a = \breve{\omega}_a + X_a Z, \tag{4.261}$$

$$\breve{\omega}_a \equiv \tilde{\omega}_a + \frac{i\tilde{u}_0}{2}\epsilon^{mn}(\psi_n\gamma_a\psi_m) + \frac{Z}{4}(\chi\gamma^3\gamma_a\gamma^b\epsilon_b{}^n\psi_n). \tag{4.262}$$

Clearly, ω_a possesses the desired properties (cf. (4.15)), but it is not the minimal covariant connection. The last term in (4.261) is a function of the target space coordinates X^I only, thus covariant by itself, which leads to the conclusion that $\breve{\omega}_a$ is the required quantity. Unfortunately, no generic prescription to construct $\breve{\omega}_a$ exists however. The rest of the procedure of Section 4.5.3 for the calculation of a superdilaton action starting with (4.206) still remains valid, but with $\breve{\omega}_a$ of (4.262) replacing $\tilde{\omega}_a$.

We point out that it is essential to have the spinor field χ^α in the multiplet; just as ϕ has been identified with the usual auxiliary field of supergravity in Section 4.6.4, we observe that general supergravity (with torsion) requires an additional auxiliary spinor field χ^α.

Spherically Reduced Gravity (SRG)

The special case (1.5) with $d = 4$ for the potentials V and Z in (4.252) yields

$$\begin{aligned} L^{\mathrm{SRG}} = \int_{\mathcal{M}} \phi d\omega + X^a De_a + \chi^\alpha D\psi_\alpha - \epsilon\left(\lambda^2 + \frac{1}{4\phi}X^a X_a + \frac{3\lambda}{32\tilde{u}_0\phi^{3/2}}\chi^2\right) \\ - \frac{1}{8\phi}X^a(\chi\gamma^3\gamma_a\gamma^b e_b\psi) + \frac{i\lambda}{4\sqrt{\phi}}(\chi\gamma^a e_a\psi) \\ - \frac{i\tilde{u}_0}{2}X^a(\psi\gamma_a\psi) - \frac{1}{2}\left(2\tilde{u}_0\lambda\sqrt{\phi} - \frac{1}{16\phi}\chi^2\right)(\psi\gamma^3\psi). \end{aligned} \tag{4.263}$$

We do not write down the supersymmetry transformations which follow from (4.255)–(4.259). We just note that our transformations δe_a and $\delta\psi_\alpha$ are different from the ones obtained in [125]. There, the supergravity multiplet is the same as in the underlying model [46], identical to the one of Section 4.6.4.

The difference is related to the use of an additional scalar superfield in [125] to construct a superdilaton action. Such an approach lies outside the scope of the present thesis, where we remain within pure gPSM without additional fields which, from the point of view of PSM are 'matter' fields.

Here, according to the general derivation of Section 4.5.3, we arrive at the superdilaton action

$$L^{\mathrm{SRG}} = L^{\mathrm{dil}} + L^{\mathrm{f}}, \tag{4.264}$$

with bosonic part (1.3) and the fermionic extension

$$\begin{aligned} L^{\mathrm{f}} = \int_{\mathcal{M}} d^2x e\Big[&\chi^\alpha \tilde{\sigma}_\alpha + i\tilde{u}_0 \left\{(\partial^n \phi) + \frac{1}{2}(\chi\gamma^3\psi^n)\right\}(\psi_n\gamma^m\psi_m) \\ &+ \frac{1}{8\phi}(\partial^n\phi)(\chi\gamma^3\gamma^m\gamma_n\psi_m) - \frac{i\lambda}{4\sqrt{\phi}}(\chi\gamma^3\gamma^m\psi_m) + \tilde{u}_0\lambda\sqrt{\phi}\,\epsilon^{mn}(\psi_n\gamma^3\psi_m) \\ &- \frac{1}{32}\chi^2\left\{\frac{1}{\phi}(\psi^n\psi_n) + \frac{1}{\phi}(\psi^n\gamma_n\gamma^m\psi_m) + \frac{3\lambda}{\tilde{u}_0\phi^{3/2}} + \epsilon^{mn}(\psi_n\gamma^3\psi_m)\right\}\Big]. \end{aligned} \tag{4.265}$$

However, as already noted in the previous section, it may be convenient to use the SUSY-covariant $\breve{\omega}_a$ (cf. (4.262)) instead of $\tilde{\omega}_a$, with the result:

$$\begin{aligned} L^{\mathrm{SRG}} = \int_{\mathcal{M}} &\phi d\breve{\omega} + \chi^\alpha \breve{D}\psi_\alpha + \frac{\epsilon}{4\phi}\Big[(\partial^n\phi)(\partial_n\phi) + (\partial^n\phi)(\chi\gamma^3\psi_n) + \frac{1}{8}\chi^2(\psi^n\psi_n)\Big] \\ - \epsilon\Big[\lambda^2 + \frac{3\lambda}{32\tilde{u}_0\phi^{3/2}}\chi^2\Big] &+ \frac{i\lambda}{4\sqrt{\phi}}(\chi\gamma^a e_a\psi) - \frac{1}{2}\Big[2\tilde{u}_0\lambda\sqrt{\phi} - \frac{1}{16\phi}\chi^2\Big](\psi\gamma^3\psi). \end{aligned} \tag{4.266}$$

Katanaev-Volovich Model (KV)

The supergravity action (4.252) for the algebra of Section 4.4.5 reads

$$\begin{aligned} L^{\mathrm{KV}} = \int_{\mathcal{M}} d^2x e\Big[&\phi R + X^a\tau_a + \chi^\alpha\sigma_\alpha + \frac{\beta}{2}\phi^2 - \Lambda + \frac{\alpha}{2}X^a X_a + \frac{1}{2}\chi^2 v_2 \\ &+ \frac{\alpha}{4}X^a(\chi\gamma_a\gamma^m\psi_m) - \frac{i\tilde{u}_0\left(\frac{\beta}{2}\phi^2 - \Lambda\right)}{2u}(\chi\gamma^m\epsilon_m{}^n\psi_n) \\ &+ \frac{i\tilde{u}_0}{2}X^a\epsilon^{mn}(\psi_n\gamma_a\psi_m) + \frac{1}{2}\left(u + \frac{\alpha}{8}\chi^2\right)\epsilon^{mn}(\psi_n\gamma^3\psi_m)\Big], \end{aligned} \tag{4.267}$$

where v_2 and u were given in (4.158) and (4.159). It is invariant under the local supersymmetry transformations (4.186) and

$$\delta X^a = -\frac{\alpha}{4}X^b(\chi\gamma_b\gamma^a\gamma^3\epsilon) + \frac{i\tilde{u}_0\left(\frac{\beta}{2}\phi^2 - \Lambda\right)}{2u}(\chi\gamma^a\epsilon), \tag{4.268}$$

$$\delta\chi^\alpha = i\tilde{u}_0 X^b(\gamma_b\epsilon)^\alpha + \left(u + \frac{\alpha}{8}\chi^2\right)(\gamma^3\epsilon)^\alpha, \tag{4.269}$$

in conjunction with the transformations

$$\delta\omega = \left[-\frac{\tilde{u}_0^2 \left(\frac{\beta}{2}\phi^2 - \Lambda\right)}{u} - \frac{\alpha u}{2} \right] (\epsilon\gamma^3\psi)$$

$$- \frac{\tilde{u}_0}{2u} \left[\beta\phi + \frac{\alpha\left(\frac{\beta}{2}\phi^2 - \Lambda\right)}{2} + \frac{\tilde{u}_0^2\left(\frac{\beta}{2}\phi^2 - \Lambda\right)^2}{u^2} \right] (\chi\gamma^b\epsilon)e_b, \qquad (4.270)$$

$$\delta e_a = i\tilde{u}_0(\epsilon\gamma_a\psi) + \frac{\alpha}{4}(\chi\gamma^3\gamma_a\gamma^b\epsilon)e_b, \qquad (4.271)$$

$$\delta\psi_\alpha = -D\epsilon_\alpha + \frac{\alpha}{4}X^a(\gamma^3\gamma_a\gamma^b\epsilon)_\alpha e_b - \frac{i\tilde{u}_0\left(\frac{\beta}{2}\phi^2 - \Lambda\right)}{2u}(\gamma^b\epsilon)_\alpha e_b + \frac{\alpha}{4}\chi_\alpha(\epsilon\gamma^3\psi) \qquad (4.272)$$

of the gauge fields.

The explicit formula for the superdilaton action may easily be obtained here, the complicated formulae, however, are not very illuminating. It turns out that $\delta\omega$ and $\delta\psi_\alpha$ contain powers u^k, $k = -3, \ldots, 1$. Therefore, singularities related to the prepotential formulae (4.144) also affect these transformations.

4.6.6 $N = (1, 0)$ Dilaton Supergravity

The case where one chiral component in χ^α (say χ^-) decouples from the theory is of special interest among the degenerate algebras of Section 4.2.2 and 4.2.2.

We adopt the solution (4.72)–(4.75) for the Poisson algebra accordingly. To avoid a cross term of the form $(\chi^-\psi_+)$ appearing in $\mathcal{P}^{a\beta}\psi_\beta e_a = (\chi F^a e_a\psi)$ we have to set $f_{(s)} = f_{(t)} = 0$. Similarly $f_{(12)} = 0$ cancels a $(\chi^-\chi^+)$-term in $\frac{1}{2}\mathcal{P}^{\alpha\beta}\psi_\beta\psi_\alpha$. Furthermore we choose $\tilde{u} = \tilde{u}_0 = \text{const}$:

$$L = \int_{\mathcal{M}} \phi d\omega + X^{++}De_{++} + X^{--}De_{--} + \chi^+D\psi_+ + \chi^-D\psi_-$$
$$+ \epsilon v + \frac{v}{2Y}X^{--}e_{--}(\chi^+\psi_+ - \chi^-\psi_-) + \frac{\tilde{u}_0}{\sqrt{2}}X^{++}\psi_+\psi_+ \qquad (4.273)$$

The chiral components χ^- and ψ_- can be set to zero consistently. The remaining local supersymmetry has one parameter ϵ_+ only:

$$\delta\phi = \frac{1}{2}(\chi^+\epsilon_+), \qquad (4.274)$$

$$\delta X^{++} = 0, \qquad \delta X^{--} = -\frac{v}{2Y}X^{--}(\chi^+\epsilon_+), \qquad (4.275)$$

$$\delta\chi^+ = \sqrt{2}\tilde{u}_0X^{++}\epsilon_+, \qquad \delta\chi^- = 0, \qquad (4.276)$$

and

$$\delta\omega = \frac{v'}{2Y} X^{--} e_{--} (\chi^+ \epsilon_+), \tag{4.277}$$

$$\delta e_{++} = -\sqrt{2}\tilde{u}_0 (\epsilon_+ \psi_+) + X_{++} \left(\frac{v}{2Y}\right)^{\cdot} X^{--} e_{--} (\chi^+ \epsilon_+), \tag{4.278}$$

$$\delta e_{--} = \frac{v}{2Y} e_{--} (\chi^+ \epsilon_+) + X_{--} \left(\frac{v}{2Y}\right)^{\cdot} X^{--} e_{--} (\chi^+ \epsilon_+), \tag{4.279}$$

$$\delta\psi_+ = -D\epsilon_+ + \frac{v}{2Y} X^{--} e_{--}, \qquad \delta\psi_- = 0. \tag{4.280}$$

4.7 Solution of the Dilaton Supergravity Model

For the dilaton prepotential supergravity model of Section 4.6.4 [129–132] the Poisson algebra was derived in Section 4.4.4. The PSM field equations (4.13) and (4.14) simplify considerably in Casimir-Darboux coordinates which can be found explicitly here, as in the pure gravity PSM. This is an improvement as compared to [132] where only the *existence* of such target coordinates was used.

We start with the Poisson tensor (4.121)–(4.123) in the coordinate system $X^I = (\phi, X^{++}, X^{--}, \chi^+, \chi^-)$. The algebra under consideration has maximal rank (2|2), implying that there is one bosonic Casimir function C. Rescaling (4.120) we choose here

$$C = X^{++} X^{--} - \frac{u^2}{2\tilde{u}_0^2} + \frac{1}{2}\chi^2 \frac{u'}{2\tilde{u}_0^2}\,. \tag{4.281}$$

In regions $X^{++} \neq 0$ we use C instead of X^{--} as coordinate in target space ($X^{--} \to C$). In regions $X^{--} \neq 0$ $X^{++} \to C$ is possible.

Treating the former case explicitly we replace $X^{++} \to \lambda = -\ln|X^{++}|$ in each of the two patches $X^{++} > 0$ and $X^{++} < 0$. This function is conjugate to the generator of Lorentz transformation ϕ (cf. (1.12))

$$\{\lambda, \phi\} = 1. \tag{4.282}$$

The functions (ϕ, λ, C) constitute a Casimir-Darboux coordinate system for the bosonic sector [57, 105, 106]. Now our aim is to decouple the bosonic sector from the fermionic one. The coordinates χ^α constitute a Lorentz spinor (cf. (4.1)). With the help of X^{++} they can be converted to Lorentz scalars, i.e. $\{\tilde{\chi}^{(\pm)}, \phi\} = 0$,

$$\chi^+ \to \tilde{\chi}^{(+)} = \frac{1}{\sqrt{|X^{++}|}} \chi^+, \qquad \chi^- \to \tilde{\chi}^{(-)} = \sqrt{|X^{++}|}\chi^-. \tag{4.283}$$

A short calculation for the set of coordinates $(\phi, \lambda, C, \tilde{\chi}^{(+)}, \tilde{\chi}^{(-)})$ shows that $\{\lambda, \tilde{\chi}^{(+)}\} = 0$ but $\{\lambda, \tilde{\chi}^{(-)}\} = -\frac{\sigma u'}{\sqrt{2}\tilde{u}_0} \tilde{\chi}^{(+)}$, where $\sigma := \operatorname{sign}(X^{++})$. The

redefinition

$$\tilde{\chi}^{(-)} \to \tilde{\tilde{\chi}}^{(-)} = \tilde{\chi}^{(-)} + \frac{\sigma u}{\sqrt{2}\tilde{u}_0}\tilde{\chi}^{(+)} \tag{4.284}$$

yields $\{\lambda, \tilde{\tilde{\chi}}^{(-)}\} = 0$ and makes the algebra block diagonal. For the fermionic sector

$$\{\tilde{\chi}^{(+)}, \tilde{\chi}^{(+)}\} = \sigma\sqrt{2}\tilde{u}_0, \tag{4.285}$$

$$\{\tilde{\tilde{\chi}}^{(-)}, \tilde{\tilde{\chi}}^{(-)}\} = \sigma\sqrt{2}\tilde{u}_0 C, \tag{4.286}$$

$$\{\tilde{\chi}^{(+)}, \tilde{\tilde{\chi}}^{(-)}\} = 0 \tag{4.287}$$

is obtained. We first assume that the Casimir function C appearing explicitly on the r.h.s. of (4.286) is invertible. Then the redefinition

$$\tilde{\tilde{\chi}}^{(-)} \to \breve{\chi}^{(-)} = \frac{1}{\sqrt{|C|}}\tilde{\tilde{\chi}}^{(-)} \tag{4.288}$$

can be made. That we found the desired Casimir-Darboux system can be seen from

$$\{\breve{\chi}^{(-)}, \breve{\chi}^{(-)}\} = s\sigma\sqrt{2}\tilde{u}_0. \tag{4.289}$$

Here $s := \mathrm{sign}(C)$ denotes the sign of the Casimir function. In fact we could rescale $\breve{\chi}^{(-)}$ and $\tilde{\chi}^{(+)}$ so as to reduce the respective right hand sides to ± 1; the signature of the fermionic 2×2 block cannot be changed however. In any case, we call the local coordinates $\bar{X}^I := (\phi, \lambda, C, \tilde{\chi}^{(+)}, \breve{\chi}^{(-)})$ Casimir-Darboux since the respective Poisson tensor is constant, which is the relevant feature here. Its non-vanishing components can be read off from (4.282), (4.285) and (4.289).

Now it is straightforward to solve the PSM field equations. Bars are used to denote the gauge fields $\bar{A}_I = (\bar{A}_\phi, \bar{A}_\lambda, \bar{A}_C, \bar{A}_{(+)}, \bar{A}_{(-)})$ corresponding to the coordinates $\bar{X}^I$. The first PSM e.o.m.s (4.13) then read

$$d\phi - \bar{A}_\lambda = 0, \tag{4.290}$$

$$d\lambda + \bar{A}_\phi = 0, \tag{4.291}$$

$$dC = 0, \tag{4.292}$$

$$d\tilde{\chi}^{(+)} + \sigma\sqrt{2}\tilde{u}_0 \bar{A}_{(+)} = 0, \tag{4.293}$$

$$d\breve{\chi}^{(-)} + s\sigma\sqrt{2}\tilde{u}_0 \bar{A}_{(-)} = 0. \tag{4.294}$$

These equations decompose in two parts (which is true also in the case of several Casimir functions). In regions where the Poisson tensor is of constant rank we obtain the statement that any solution of (4.13) and (4.14) lives on symplectic leaves which is expressed here by the differential equation

(4.292) with the one-parameter solution $C = c_0 =$ const. Eqs. (4.290)–(4.294) without (4.292) are to be used to solve for all gauge fields excluding the ones which correspond to the Casimir functions, thus excluding $\bar{A}_C$ in our case. Note that this solution is purely algebraic. The second set of PSM equations (4.14) again split in two parts. The equations $d\bar{A}_\phi = d\bar{A}_\lambda = d\bar{A}_{(+)} = d\bar{A}_{(-)} = 0$ are identically fulfilled, as can be seen from (4.290)–(4.294). To show this property in the generic case the first PSM equations (4.13) in conjunction with the Jacobi identity of the Poisson tensor have to be used. The remainder of the second PSM equations are the equations for the gauge fields corresponding to the Casimir functions. In a case as simple as ours we find, together with the local solution in terms of an integration function $F(x)$ (taking values in the commuting supernumbers),

$$d\bar{A}_C = 0 \quad \Rightarrow \quad \bar{A}_C = -dF. \tag{4.295}$$

The explicit solution for the original gauge fields $A_I = (\omega, e_a, \psi_\alpha)$ is derived from the target space transformation $A_I = (\partial_I \bar{X}^J)\bar{A}_J$, but in order to compare with the case $C = 0$ we introduce an intermediate step and give the solution in coordinates $\tilde{X}^I = (\phi, \lambda, C, \tilde{\chi}^{(+)}, \tilde{\tilde{\chi}}^{(-)})$ first. With $\tilde{A}_I = (\tilde{A}_\phi, \tilde{A}_\lambda, \tilde{A}_C, \tilde{A}_{(+)}, \tilde{A}_{(-)})$ the calculation $\tilde{A}_I = (\tilde{\partial}_I \bar{X}^J)\bar{A}_J$ yields

$$\tilde{A}_\phi = -d\lambda, \qquad \tilde{A}_\lambda = d\phi, \qquad \tilde{A}_{(+)} = -\frac{\sigma}{\sqrt{2}\tilde{u}_0} d\tilde{\chi}^{(+)}, \tag{4.296}$$

and

$$\tilde{A}_C = -dF + \frac{\sigma}{2\sqrt{2}\tilde{u}_0 C^2} \tilde{\tilde{\chi}}^{(-)} d\tilde{\tilde{\chi}}^{(-)}, \qquad \tilde{A}_{(-)} = -\frac{\sigma}{\sqrt{2}\tilde{u}_0 C} d\tilde{\tilde{\chi}}^{(-)}. \tag{4.297}$$

This has to be compared with the case $C = 0$. Obviously, the fermionic sector is no longer of full rank, and $\tilde{\tilde{\chi}}^{(-)}$ is an additional, fermionic Casimir function as seen from (4.286) and also from the corresponding field equation

$$d\tilde{\tilde{\chi}}^{(-)} = 0. \tag{4.298}$$

Thus, the $\tilde{X}^I$ are Casimir-Darboux coordinates on the subspace $C = 0$. The PSM e.o.m.s in barred coordinates still are of the form (4.290)–(4.293). Therefore, the solution (4.296) remains unchanged, but (4.297) has to be replaced by the solution of $d\tilde{A}_C = 0$ and $d\tilde{A}_{(-)} = 0$. In terms of the bosonic function $F(x)$ and an additional fermionic function $\rho(x)$ the solution is

$$\tilde{A}_C = -dF, \qquad \tilde{A}_{(-)} = -d\rho. \tag{4.299}$$

Collecting all formulae the explicit solution for the original gauge fields $A_I = (\omega, e_a, \psi_\alpha)$ calculated with $A_I = (\partial_I \tilde{X}^J)\tilde{A}_J$ reads (cf. Appendix B.1

for the definition of ++ and −− components of Lorentz vectors)

$$\omega = \frac{dX^{++}}{X^{++}} + V\tilde{A}_C + \frac{\sigma u'}{\sqrt{2}\tilde{u}_0}\tilde{\chi}^{(+)}\tilde{A}_{(-)}, \tag{4.300}$$

$$e_{++} = -\frac{d\phi}{X^{++}} + X^{--}\tilde{A}_C + \frac{1}{2X^{++}}\left[\frac{\sigma}{\sqrt{2}\tilde{u}_0}\tilde{\chi}^{(+)}d\tilde{\chi}^{(+)} + \left(\tilde{\chi}^{(-)} - \frac{\sigma u}{\sqrt{2}\tilde{u}_0}\tilde{\chi}^{(+)}\right)\tilde{A}_{(-)}\right], \tag{4.301}$$

$$e_{--} = X^{++}\tilde{A}_C, \tag{4.302}$$

$$\psi_+ = -\frac{u'}{2\tilde{u}_0^2}\chi^-\tilde{A}_C - \frac{\sigma}{\sqrt{2}\tilde{u}_0}\frac{1}{\sqrt{|X^{++}|}}\left(d\tilde{\chi}^{(+)} - u\tilde{A}_{(-)}\right), \tag{4.303}$$

$$\psi_- = \frac{u'}{2\tilde{u}_0^2}\chi^+\tilde{A}_C + \sqrt{|X^{++}|}\tilde{A}_{(-)}. \tag{4.304}$$

$\tilde{A}_C$ and $\tilde{A}_{(-)}$ are given by (4.297) for $C \neq 0$ and by (4.299) for $C = 0$.

For $C \neq 0$ our solution depends on the free function F and the coordinate functions $(\phi, X^{++}, X^{--}, \chi^+, \chi^-)$ which, however, are constrained by $C = c_0 = \text{const}$ according to (4.281). For $C = 0$ the free functions are F and ρ. The coordinate functions $(\phi, X^{++}, X^{--}, \chi^+, \chi^-)$ here are restricted by $C = 0$ in (4.281) and by $\tilde{\tilde{\chi}}^{(-)} = \text{const}$.

This solution holds for $X^{++} \neq 0$; an analogous set of relations can be derived exchanging the role of X^{++} and X^{--}.

The solution (cf. (4.301) and (4.302)) is free from coordinate singularities in the line element, exhibiting a sort of 'super Eddington-Finkelstein' form. For special choices of the potentials $v(\phi)$ or the related prepotential $u(\phi)$ we refer to Table 4.1.

This provides also the solution for the models with quadratic bosonic torsion of Section 4.6.5 by a further change of variables (4.129) with parameter (4.143). Its explicit form is calculated using (4.139).

As of now we did not use the gauge freedom. Actually, in supergravity theories this is generically not that easy, since the fermionic part of the symmetries are known only in their *infinitesimal* form (the bosonic part corresponds on a global level to diffeomorphisms and local Lorentz transformations). This changes in the present context for the case that Casimir-Darboux coordinates are available. Indeed, for a constant Poisson tensor the otherwise field-dependent, nonlinear symmetries (4.15) can be integrated easily: Within the range of applicability of the target coordinates, $\bar{X}^I$ may be shifted by some arbitrary function (except for the Casimir function C, which, however, was found to be constant over $\mathcal{M}$), and $\bar{A}_C$ may be redefined by the addition of some exact part. The only restrictions to these symmetries come from nondegeneracy of the metric (thus e.g. $\tilde{A}_C$ should not be put to zero, cf. (4.302)). In particular we are thus allowed to put both $\tilde{\chi}^{(+)}$ and $\breve{\chi}^{(-)}$ to zero in the present patch, and thus, if one follows back the

transformations introduced, also the original fields $\chi^{\pm}$. Next, in the patches with $X^{++} \neq 0$, we may fix the local Lorentz invariance by $X^{++} := 1$ and the diffeomorphism invariance by choosing ϕ and F as local coordinates on the spacetime manifold $\mathcal{M}$. The resulting gauge fixed solution agrees with the one found in the original bosonic theory, cf. e.g. [59]. This is in agreement with the general considerations of [132], here however made explicit.

A final remark concerns the discussion following (4.124): As noted there, for some choices of the bosonic potential v the potential u becomes complex valued if the range of ϕ is not restricted appropriately. It is straightforward to convince oneself that the above formulae are still valid in the case of complex valued potentials u (i.e. complex valued Poisson tensors). Just in intermediary steps, such as (4.284), one uses complex valued fields (with some reality constraints). The final gauge fixed solution, however, is not affected by this, being real as it should be.

Chapter 5

PSM with Superfields

So far the question is still open how the supergravity models derived using the graded PSM approach (cf. Chapter 4 above) fit within the superfield context. The former approach relies on a Poisson tensor of a target space, but in the superfield context of Chapter 2 there was no such structure, even worse, there was no target space at all. The connection can be established by extending the gPSM to superspace.

5.1 Outline of the Approach

5.1.1 Base Supermanifold

It is suggestive to enlarge the two-dimensional base manifold $\mathcal{M}$ with coordinates x^m to become a four-dimensional supermanifold $S_{\mathcal{M}}$ parametrized by coordinates $z^M = (x^m, \theta^\mu)$. We do not modify the target space $\mathcal{N}$, where we retain still the coordinates $X^I = (\phi, X^a, \chi^\alpha)$ and the Poisson tensor $\mathcal{P}^{IJ}(X)$ as in Chapter 4. When we consider the map $S_{\mathcal{M}} \to \mathcal{N}$ in coordinate representation, we obtain the superfields $X^I(z) = X^I(x, \theta)$, which we expand as usual

$$X^I(z) = X^I(x) + \theta^\mu X_\mu{}^I(x) + \frac{1}{2}\theta^2 X_2{}^I(x). \tag{5.1}$$

It is important to note that there is no change of the Poisson tensor of the model. If we want to calculate the equations of the PSM apparatus the only thing we have to do with the Poisson tensor is to evaluate it at $X(z)$, i.e. $\mathcal{P}^{IJ}(X(z))$.

In the PSM with base manifold $\mathcal{M}$ we had the gauge fields A_I. These former 1-forms on $\mathcal{M}$ are now promoted to 1-superforms on $S_{\mathcal{M}}$,

$$A_I(z) = dz^M A_{MI}(z). \tag{5.2}$$

The enlargement of the base manifold thus added further gauge fields to the model. Beside the already known ones,

$$A_{mI} = (\omega_m, e_{ma}, \psi_{m\alpha}), \tag{5.3}$$

the new fields

$$A_{\mu I} = (\omega_\mu, e_{\mu a}, \psi_{\mu\alpha}) \tag{5.4}$$

are obtained.

Because we don't know the PSM action for base manifolds of dimension other than two, nor do we know the action for base supermanifolds, the further analysis rests solely on the properties of the PSM equations of motion. The observation that neither the closure of the PSM field equations nor its symmetries depend on the underlying base manifold but only on the Jacobi identities of the Poisson tensor makes this possible. The field equations in superspace read

$$dX^I + \mathcal{P}^{IJ} A_J = 0, \tag{5.5}$$

$$dA_I + \frac{1}{2}(\partial_I \mathcal{P}^{JK}) A_K A_J = 0, \tag{5.6}$$

and the symmetry transformations with parameters $(\epsilon_I) = (l, \epsilon_a, \epsilon_\alpha)$, which are now functions on $S_\mathcal{M}$, $\epsilon_I = \epsilon_I(z)$, are given by

$$\delta X^I = \mathcal{P}^{IJ} \epsilon_J, \tag{5.7}$$

$$\delta A_I = -d\epsilon_I - (\partial_I \mathcal{P}^{JK}) \epsilon_K A_J. \tag{5.8}$$

Collecting indices by introducing a superindex $A = (a, \alpha)$ the supervielbein

$$E_{MA} = \begin{pmatrix} e_{ma} & \psi_{m\alpha} \\ e_{\mu a} & \psi_{\mu\alpha} \end{pmatrix} \tag{5.9}$$

can be formed. The vielbein $e_m{}^a$ therein is invertible per definition and we assume further that $\psi_\mu{}^\alpha$ possesses this property too. Lorentz indices are raised and lowered with metric η_{ab}, spinor indices with $\epsilon_{\alpha\beta}$ so that the space spanned by the coordinates X^A is equipped with the direct product supermetric

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \epsilon_{\alpha\beta} \end{pmatrix}. \tag{5.10}$$

This allows to raise and lower A-type indices, $E_M{}^A = \eta^{AB} E_{MB}$. The inverse supervielbein fulfills $E_A{}^M E_M{}^B = \delta_A{}^B$.

5.1.2 Superdiffeomorphism

Our next task is to recast superdiffeomorphism of the base manifold $S_{\mathcal{M}}$ into PSM symmetry transformations with parameter ϵ_I. Let

$$\delta z^M = -\xi^M(z) \tag{5.11}$$

be the infinitesimal displacement of points on $S_{\mathcal{M}}$ by a vector superfield

$$\xi^M(z) = \xi^M(x) + \theta^\mu \xi_\mu{}^M(x) + \frac{1}{2}\theta^2 \xi_2{}^M(x). \tag{5.12}$$

The transformations of the fields $X^I(z)$ under (5.11),

$$\delta X^I(z) = -\xi^M(z)\partial_M X^I(z), \tag{5.13}$$

can be brought to the form (5.7) by taking the field equations (5.5) into account, in components they read $\partial_M X^I = -\mathcal{P}^{IJ} A_{MJ}(-1)^{M(I+J)}$, and by the choice

$$\epsilon_I(z) = \xi^M(z) A_{MI}(z) \tag{5.14}$$

for the PSM symmetry parameters. To confirm (5.14) the transformations δA_I have to be considered in order to rule out the arbitrariness $\epsilon_I \to \epsilon_I + \partial_I C$ which does not change the transformation rules (5.7) of X^I.

If we want to go the other way and choose $\epsilon_A(z)$ as the primary transformation parameters, the corresponding $\xi^M(z)$ can be derived from $\epsilon_A(z) = \xi^M(z) E_{MA}(z)$ due to the invertibility of the supervielbein, but note that we have to accommodate for an additional $\epsilon_A(z)$-dependent Lorentz transformation in order to recast superdiffeomorphism:

$$\xi^M(\epsilon_A(z); z) = \epsilon^A(z) E_A{}^M(z), \tag{5.15}$$

$$l(\epsilon_A(z); z) = \epsilon^A(z) E_A{}^M(z)\omega_M(z). \tag{5.16}$$

5.1.3 Component Fields

The component fields of supergravity in the superfield formulation can be restored by solving the PSM field equations partly. This can be achieved by separating the ∂_ν derivatives in the PSM field equations (5.5) and (5.6). From (5.5)

$$\partial_\nu X^I(z) + \mathcal{P}^{IJ}(X(z)) A_{\nu J}(z)(-1)^{I+J} = 0 \tag{5.17}$$

is obtained. To zeroth order in θ this equation expresses $X_\mu{}^I(x)$ (cf. (5.1)) in terms of $X^I(x)$ and $A_{\nu J}(x) = A_{\nu J}(x, 0)$. For the same purpose we choose

$$\partial_\nu A_{\mu I}(z) + \partial_\mu A_{\nu I}(z) - \partial_I \mathcal{P}^{JK}|_{X(z)} A_{\mu K}(z) A_{\nu J}(z)(-1)^K = 0, \tag{5.18}$$

$$\partial_\nu A_{mI}(z) + \partial_m A_{\nu I}(z) + \partial_I \mathcal{P}^{JK}|_{X(z)} A_{mK}(z) A_{\nu J}(z)(-1)^{I+J} = 0 \tag{5.19}$$

from the field equations (5.6). The equations (5.17), (5.18) and (5.19) can be solved simultaneously order by order in θ, therefore expressing higher order θ-components of $X^I(z)$, $A_{mI}(z)$ and $A_{\mu I}(z)$ in terms of $X^I(x)$, $A_{mI}(x)$, $A_{\mu I}(x)$, $(\partial_{[\nu}A_{\mu]I})(x)$ and $(\partial_m A_{\mu I})(x)$, $(\partial_m\partial_{[\nu}A_{\mu]I})(x)$.

Counting one x-space field as one degree of freedom there are 16 independent symmetry parameters in the vector superfield $\xi^M(z)$, the same gauge degree of freedom as in $\epsilon_A(z) = (\epsilon_a(z), \epsilon_\alpha(z))$, and there are 4 independent parameters in the Lorentz supertransformation $l(z) = \epsilon_\phi(z)$. In summary 20 gauge degrees of freedom are found in the set of superfield parameters $\epsilon_I(z) = (l(z), \epsilon_a(z), \epsilon_\alpha(z))$. This huge amount of x-space symmetries can be broken by gauging some component fields of the newly added fields (5.4). In particular we impose the conditions

$$\omega_\mu|_{\theta=0} = 0, \qquad e_{\mu a}|_{\theta=0} = 0, \qquad \psi_{\mu\alpha}|_{\theta=0} = \epsilon_{\mu\alpha}, \tag{5.20}$$

$$\partial_{[\nu}\omega_{\mu]}|_{\theta=0} = 0, \qquad \partial_{[\nu}e_{\mu]a}|_{\theta=0} = 0, \qquad \partial_{[\nu}\psi_{\mu]\alpha}|_{\theta=0} = 0. \tag{5.21}$$

This gauge conditions turn out to be identical to the ones of the superfield formulation cf. Section 2.3.2 and 2.5.3.

Counting the number of gauged x-space fields we obtain 10 for (5.20) and 5 for (5.21), thus reducing the number of independent x-field symmetry parameters to 5. These transformations are given by $\epsilon_I(x) = \epsilon_I(x, \theta = 0)$, where local Lorentz rotation, local translation and local supersymmetry can be identified within the parameter set $\epsilon_I(x) = (l(x), \epsilon_a(x), \epsilon_\alpha(x))$. Alternatively, one could use $\xi^M(x)$ instead of $\epsilon_A(x)$ (cf. Section 5.1.2 before) to account for translation and supersymmetry. All θ-components of $\epsilon_I(z)$ can be determined from the PSM symmetries (5.8),

$$\delta A_{\mu I}(z) = -\partial_\mu \epsilon_I(z) - \partial_I \mathcal{P}^{JK}|_{X(z)}\epsilon_K A_{\mu J}(-1)^{I+J}, \tag{5.22}$$

when the gauge fixing conditions (5.20) and (5.21) for $A_{\mu I}$ are taken into account.

5.2 Rigid Supersymmetry

The Poisson tensor of rigid supersymmetry is

$$\mathcal{P}^{a\phi} = X^b \epsilon_b{}^a, \qquad \mathcal{P}^{\alpha\phi} = -\frac{1}{2}\chi^\beta \gamma^3{}_\beta{}^\alpha, \tag{5.23}$$

$$\mathcal{P}^{\alpha\beta} = 2i\tilde{c}X^a \gamma_a{}^{\alpha\beta} + 2c\gamma^{3\alpha\beta}, \tag{5.24}$$

$$\mathcal{P}^{ab} = 0, \qquad \mathcal{P}^{a\beta} = 0, \tag{5.25}$$

where c and $\tilde{c}$ are constant.

Choosing the gauge conditions (5.20) and (5.21) we can solve (5.17) for $X^I(z)$ at once. In order to make notations more transparent we drop the

arguments of the functions and place a hat above a symbol, if it is meant to be a function of z, otherwise with no hat it is a function of x only. We get

$$\hat{\phi} = \phi + \frac{1}{2}(\theta\gamma^3\chi) - \frac{1}{2}\theta^2 c, \tag{5.26}$$

$$\hat{X}^a = X^a, \tag{5.27}$$

$$\hat{\chi}^\alpha = \chi^\alpha + 2i\tilde{c}X^a(\theta\gamma_a)^\alpha + 2c(\theta\gamma^3)^\alpha. \tag{5.28}$$

Solving (5.18) yields $A_{\mu I}(z)$,

$$\hat{\omega}_\mu = 0, \tag{5.29}$$

$$\hat{e}_{\mu a} = -i\tilde{c}(\theta\gamma_a)_\mu, \tag{5.30}$$

$$\hat{\psi}_{\mu\alpha} = \epsilon_{\mu\alpha}, \tag{5.31}$$

and finally we get the component fields of $A_{mI}(z)$ using (5.19),

$$\hat{\omega}_m = \omega_m, \tag{5.32}$$

$$\hat{e}_{ma} = e_{ma} - 2i\tilde{c}(\theta\gamma_a\psi_m), \tag{5.33}$$

$$\hat{\psi}_{m\alpha} = \psi_{m\alpha} - \frac{1}{2}\omega_m(\theta\gamma^3)_\alpha. \tag{5.34}$$

The inverse of the supervielbein $\hat{E}_M{}^A$, cf. (5.9), is given by $\hat{E}_A{}^M$ with the components

$$\hat{E}_a{}^m = e_a{}^m + i\tilde{c}(\theta\gamma^m\psi_a) - \frac{1}{2}\theta^2\,\tilde{c}^2(\psi_a\gamma^n\gamma^m\psi_n), \tag{5.35}$$

$$\begin{aligned}\hat{E}_a{}^\mu = &-\psi_a{}^\mu - i\tilde{c}(\theta\gamma^n\psi_a)\psi_n{}^\mu + \frac{1}{2}\omega_a(\theta\gamma^3)^\mu \\ &+ \frac{1}{2}\theta^2\left[\tilde{c}^2(\psi_a\gamma^n\gamma^m\psi_n)\psi_m{}^\mu + \frac{i}{2}\tilde{c}\omega_n(\psi_a\gamma^n\gamma^3)^\mu\right],\end{aligned} \tag{5.36}$$

$$\hat{E}_\alpha{}^m = i\tilde{c}(\theta\gamma^m)_\alpha - \frac{1}{2}\theta^2\,\tilde{c}^2(\gamma^n\gamma^m\psi_n)_\alpha, \tag{5.37}$$

$$\hat{E}_\alpha{}^\mu = \delta_\alpha{}^\mu - i\tilde{c}(\theta\gamma^n)_\alpha\psi_n{}^\mu + \frac{1}{2}\theta^2\left[\tilde{c}^2(\gamma^n\gamma^m\psi_n)_\alpha\psi_m{}^\mu + \frac{i}{2}\tilde{c}\omega_n(\gamma^n\gamma^3)_\alpha{}^\mu\right]. \tag{5.38}$$

The chosen gauge restricts the symmetry parameters $\epsilon_I(z)$ due to the variations (5.22), from which we get

$$\hat{l} = l, \tag{5.39}$$

$$\hat{\epsilon}_a = \epsilon_a + 2i\tilde{c}(\epsilon\gamma_a\theta), \tag{5.40}$$

$$\hat{\epsilon}_\alpha = \epsilon_\alpha - \frac{1}{2}l(\theta\gamma^3)_\alpha. \tag{5.41}$$

Using (5.15) and (5.16) we calculate the world sheet transformations corresponding to above parameters. For parameter $\epsilon_a(x)$ only, we get

$$\hat{\xi}^m = \epsilon^a \hat{E}_a{}^m, \qquad \hat{\xi}^\mu = \epsilon^a \hat{E}_a{}^\mu, \qquad \hat{l} = \epsilon^a \hat{E}_a{}^m \omega_m. \tag{5.42}$$

$$\hat{\xi}^m = \epsilon^a e_a{}^m - i\tilde{c}\epsilon^a(\psi_a\gamma^m\theta) - \frac{1}{2}\theta^2\,\tilde{c}^2\epsilon^a(\psi_a\gamma^n\gamma^m\psi_n), \tag{5.43}$$

$$\begin{aligned}\hat{\xi}^\mu = {}& \frac{1}{2}\epsilon^a\omega_a(\theta\gamma^3)^\mu - \epsilon^a\psi_a{}^\mu + i\tilde{c}\epsilon^a(\psi_a\gamma^n\theta)\psi_n{}^\mu \\ & + \frac{1}{2}\theta^2\left[\tilde{c}^2\epsilon^a(\psi_a\gamma^n\gamma^m\psi_n)\psi_m{}^\mu + \frac{i}{2}\tilde{c}\omega_n\epsilon^a(\psi_a\gamma^n\gamma^3)^\mu\right],\end{aligned} \tag{5.44}$$

$$\hat{l} = \epsilon^a\omega_a - i\tilde{c}\epsilon^a(\psi_a\gamma^m\theta)\omega_m - \frac{1}{2}\theta^2\,\tilde{c}^2\epsilon^a(\psi_a\gamma^n\gamma^m\psi_n)\omega_m. \tag{5.45}$$

Making a supersymmetry transformation with parameter $\epsilon_\alpha(x)$ yields

$$\hat{\xi}^m = i\tilde{c}(\epsilon\gamma^m\theta) + \frac{1}{2}\theta^2\,\tilde{c}^2(\epsilon\gamma^n\gamma^m\psi_n), \tag{5.46}$$

$$\hat{\xi}^\mu = \epsilon^\mu - i\tilde{c}(\epsilon\gamma^n\theta)\psi_n{}^\mu + \frac{1}{2}\theta^2\left[-\tilde{c}^2(\epsilon\gamma^n\gamma^m\psi_n)\psi_m{}^\mu - \frac{i}{2}\tilde{c}\omega_n(\epsilon\gamma^n\gamma^3)^\mu\right], \tag{5.47}$$

$$\hat{l} = i\tilde{c}(\epsilon\gamma^m\theta)\omega_m + \frac{1}{2}\theta^2\,\tilde{c}^2(\epsilon\gamma^n\gamma^m\psi_n)\omega_m. \tag{5.48}$$

Finally, for Lorentz transformations with parameter $l(x)$ the correct representation in superspace is

$$\hat{\xi}^m = 0, \qquad \hat{\xi}^\mu = -\frac{1}{2}l(\theta\gamma^3)^\mu, \qquad \hat{l} = l. \tag{5.49}$$

When analyzing (5.43)–(5.45) we recognize that a $\epsilon_a(x)$-transformation creates a pure bosonic diffeomorphism on the world sheet, $\delta x^m = -\eta^m(x)$, with parameter $\eta^m(x) = \epsilon^a(x)e_a{}^m(x)$, a Lorentz transformation with parameter $l(x) = \epsilon^a(x)\omega_a(x)$ and a supersymmetry transformation with parameter $\epsilon^\alpha(x) = -\epsilon^a(x)\psi_a{}^\alpha(x)$. We can go the other way and impose a bosonic diffeomorphism $\delta x^m = -\eta^m(x)$. Then using $\hat{\epsilon}_I = \eta^m\hat{A}_{mI}$ (cf. (5.14)) we obtain

$$\hat{l} = \eta^m\omega_m, \tag{5.50}$$

$$\hat{\epsilon}_a = \eta^m e_{ma} + 2i\tilde{c}\eta^m(\psi_m\gamma_a\theta), \tag{5.51}$$

$$\hat{\epsilon}_\alpha = \eta^m\psi_{m\alpha} - \frac{1}{2}\eta^m\omega_m(\theta\gamma^3)_\alpha, \tag{5.52}$$

where we see that we not only get an ϵ_a-transformation but also a Lorentz and a supersymmetry transformation.

Finally we note the transformations of the scalars $X^I(x)$,

$$\delta\phi = -X^b \epsilon_b{}^a \epsilon_a - \frac{1}{2}(\epsilon\gamma^3\chi), \tag{5.53}$$

$$\delta X^a = l X^b \epsilon_b{}^a, \tag{5.54}$$

$$\delta\chi^\alpha = -\frac{1}{2} l (\chi\gamma^3)^\alpha - 2i\tilde{c}X^a(\epsilon\gamma_a)^\alpha - 2c(\epsilon\gamma^3)^\alpha, \tag{5.55}$$

and the transformations of the gauge fields $A_{mI}(x)$,

$$\delta\omega_m = -\partial_m l, \tag{5.56}$$

$$\delta e_{ma} = -D_m \epsilon_a - l\epsilon_a{}^b e_{mb} + 2i\tilde{c}(\epsilon\gamma_a\psi_m), \tag{5.57}$$

$$\delta\psi_{m\alpha} = -D_m\epsilon_\alpha + \frac{1}{2} l (\gamma^3\psi_m)_\alpha. \tag{5.58}$$

We can see that there is nothing new, because this is just the result that we had for a PSM with the purely bosonic base manifold $\mathcal{M}$. The same is true for the remaining superspace PSM equations. They are just the same as in the simpler model with world sheet $\mathcal{M}$.

5.3 Supergravity Model of Howe

In order to establish the connection between the superfield method of Chapter 2 and the graded PSM approach of Chapter 4 explicitly we choose the well-known supergravity model of Howe [46]. It was derived with superfield methods, further developments can be found in [138–141]. For the purposes needed here a summary of the superfield expressions for that model was given in Section 2.3.

The corresponding Poisson tensor for the Howe model has been derived in Chapter 4:

$$\mathcal{P}^{\alpha\beta} = i\tilde{u}_0 X^a \gamma_a{}^{\alpha\beta} + \tilde{u}_0 \lambda\phi^2 \gamma^{3\alpha\beta} \tag{5.59}$$

$$\mathcal{P}^{\alpha b} = i\lambda\phi(\chi\gamma^b)^\alpha \tag{5.60}$$

$$\mathcal{P}^{ab} = \left(-2\lambda^2\phi^3 + \frac{\lambda}{2\tilde{u}_0}\chi^2\right)\epsilon^{ab} \tag{5.61}$$

Here λ and $\tilde{u}_0$ are constant. For $\mathcal{P}^{a\phi}$ and $\mathcal{P}^{\alpha\phi}$ we refer to (4.25) in Section 4.2.1, where also the ansatz of the Poisson tensor (4.26)–(4.34) defining the various potentials can be found. The Poisson tensor was derived in Section 4.4.4, in particular the prepotential $u(\phi)$ was given in Table 4.1. The corresponding PSM action and its symmetries are to be found in Section 4.6.4.

With the Wess-Zumino type gauge conditions (5.20) and (5.21) the PSM field equation (5.17) can be solved for $X^I(z)$ at once. In order to make

notations more concise the arguments of the functions are dropped and replaced by a hat above a symbol to signalize that it is a function of z, otherwise with no hat it is a function of x only. We obtain

$$\hat{\phi} = \phi + \frac{1}{2}(\theta\gamma^3\chi) - \frac{\lambda\tilde{u}_0}{4}\theta^2\phi^2, \tag{5.62}$$

$$\hat{X}^a = X^a - i\lambda\phi(\theta\gamma^a\chi) - \frac{\lambda\tilde{u}_0}{2}\theta^2\phi X^a, \tag{5.63}$$

$$\hat{\chi}^\alpha = \chi^\alpha + i\tilde{u}_0 X^a(\theta\gamma_a)^\alpha + \lambda\tilde{u}_0\phi^2(\theta\gamma^3)^\alpha + \frac{3\lambda\tilde{u}_0}{4}\theta^2\phi\chi^\alpha. \tag{5.64}$$

Solving (5.18) yields $A_{\mu I}(z)$,

$$\hat{\omega}_\mu = -\lambda\tilde{u}_0\phi(\theta\gamma^3)_\mu, \tag{5.65}$$

$$\hat{e}_{\mu a} = -\frac{i\tilde{u}_0}{2}(\theta\gamma_a)_\mu, \tag{5.66}$$

$$\hat{\psi}_{\mu\alpha} = \left(1 + \frac{\lambda\tilde{u}_0}{4}\theta^2\phi\right)\epsilon_{\mu\alpha}, \tag{5.67}$$

and $A_{mI}(z)$ derived from (5.19) reads,

$$\begin{aligned}\hat{\omega}_m = \omega_m &+ i\lambda(\theta\gamma_m\chi) - 2\lambda\tilde{u}_0\phi(\theta\gamma^3\psi_m) \\ &+ \frac{\lambda\tilde{u}_0}{2}\theta^2\left[X_m - \phi\omega_m + \frac{1}{2}(\chi\psi_m)\right],\end{aligned} \tag{5.68}$$

$$\hat{e}_{ma} = e_{ma} - i\tilde{u}_0(\theta\gamma_a\psi_m) - \frac{\lambda\tilde{u}_0}{2}\theta^2\phi e_{ma}, \tag{5.69}$$

$$\begin{aligned}\hat{\psi}_{m\alpha} = \psi_{m\alpha} &- \frac{1}{2}\omega_m(\theta\gamma^3)_\alpha + i\lambda\phi(\theta\gamma_m)_\alpha \\ &+ \frac{1}{2}\theta^2\left[\frac{3\lambda\tilde{u}_0}{2}\phi\psi_{m\alpha} - \frac{i\lambda}{2}(\chi\gamma_m\gamma^3)_\alpha\right].\end{aligned} \tag{5.70}$$

The choice $\lambda = \frac{1}{2}$ and $\tilde{u}_0 = -2$ together with the identification of the dilaton as the supergravity auxiliary field $\phi = A$ (cf. also Section 4.6.4) should lead to the superfield expressions obtained in Section 2.3.1. Indeed, (5.65), (5.66) and (5.67) are already identical to the formulae (2.105), (2.90) and (2.91), respectively. The same is true for (5.69) as a comparison to (2.88) confirms, but X_m and χ^α are present in (5.68) and (5.70) which do not show up directly in (2.104) and (2.89). Furthermore ω_m in (5.68) and (5.70) is an independent x-space field, but in supergravity from superspace only the dependent connection $\breve{\omega}_m$ (cf. (2.84)) is left over.

First we take a closer look at ω_m and X_m: To zeroth order in θ the superspace field equations (5.5) and (5.6) reduce to the ones of graded PSM where the underlying base manifold is two-dimensional, so that the considerations of Chapter 4 apply. Especially we refer to Section 4.5.3 where a uniform way to eliminate ω_m and X^a at once was offered. From (4.205) for

the class of dilaton prepotential supergravities (cf. Section 4.6.4) to which the Poisson tensor considered here belongs the result was already derived and given in (4.239). This is identical to (2.84) as required. For X^a the expression (4.211) was obtained, which does not rely on a particular Poisson tensor at all.

Once the equivalence $\omega \equiv \breve{\omega}$ is established and after replacing X^a in (5.68) by (4.211) it remains to show that

$$\chi_\alpha = -\frac{\tilde{u}_0}{\lambda}\breve{\sigma}_\alpha, \tag{5.71}$$

where $\breve{\sigma}_\alpha$ was given by (2.86). The reader should be aware of the fact that $\breve{\sigma}_\alpha$ was defined without the auxiliary scalar field in Chapter 4, cf. also (4.201) and (4.251). Actually (5.71) is a PSM field equation. To zeroth order in θ in (5.6) the x-space differential form equation

$$D\psi_\alpha + i\lambda\phi(\gamma^a e_a \psi)_\alpha + \frac{\lambda}{\tilde{u}_0}\epsilon\chi_\alpha = 0 \tag{5.72}$$

is contained, from which (5.71) is derived.

To summarize: with $\omega \equiv \breve{\omega}$ (cf. (4.239)), X^a given by (4.211), and with χ_α as in (5.71) the superconnection (5.68) and the Rarita-Schwinger superfield (5.70) become to be identical to the ones obtained from superspace constraints namely (2.104) and (2.89).

Chapter 6

Conclusion

In Chapter 2 a new formulation of the superfield approach to general supergravity has been established which allows nonvanishing bosonic torsion. It is based upon the new minimal set of constraints (2.156) for $N = (1, 1)$ superspace. The computational problems which would occur for nonvanishing torsion following the approach of the seminal work of Howe [46] are greatly reduced by working in terms of a special decomposition of the supervielbein in terms of superfields $B_m{}^a$, $B_\mu{}^\alpha$, $\Phi_\mu{}^m$ and $\Psi_m{}^\mu$ (cf. (2.69) and (2.70)).

In Section 2.5.4 the component fields in the Wess-Zumino type gauge (2.189) and (2.190) were derived explicitly for the decompositon superfields as well as for the supervielbein and the Lorenz superconnection. They also turned out to be expressed by an additional, new multiplet $\{k^a, \varphi_m{}^\alpha, \omega_m\}$ consisting of a vector field, a spin-vector and the Lorentz connection, besides the well-known supergravity multiplet $\{e_m{}^a, \psi_m{}^\alpha, A\}$ consisting of the zweibein, the Rarita-Schwinger field and a scalar A.

The Bianchi identities have shown that the scalar superfield S, already present in the Howe model, is accompanied by a vector superfield K^a. Together they represent the independent and unconstrained superfields of supertorsion and supercurvature (cf. (2.173)–(2.177)). Since the component fields transform with respect to the correct local diffeomorphisms, local Lorentz and supersymmetry transformations, a generic superfield Lagrangian is a superscalar built from S and K^a, multiplied by the superdeterminant of $E_M{}^A$. With that knowledge one could produce the immediate generalization of two-dimensional gravity with torsion [54, 55], but the explicit derivation of the component field content of supertorsion and supercurvature turned out to be extremely cumbersome. That goal may be attained with the help of a computer algebra program [142] in future investigations.

In the course of introducing the PSM concept we also presented (Chapter 3) a new extension of that approach, where the singular Poisson structure of the PSM is embedded into a symplectic one. In this connection we also showed how the general solution of the field equations can be obtained in a

systematic manner.

In Chapter 4 the extension of the concept of Poisson Sigma Models (PSM) to the graded case [129, 130, 132] has been explored in some detail for the application in general two-dimensional supergravity theories, when a dilaton field is present. Adding one ($N = 1$) or more ($N > 1$) pairs of Majorana fields representing respectively a target space (spinor) variable χ^α and a related 'gravitino' $\psi_m{}^\alpha$, automatically leads to a supergravity with local supersymmetry closing on-shell. Our approach yields the minimal supermultiplets, avoiding the imposition and evaluation of constraints which is necessary in the superfield formalism. Instead we have to solve Jacobi identities, which the (degenerate) Poisson structure $\mathcal{P}^{AB}$ of a PSM must obey. In our present work we have performed this task for the full $N = 1$ problem. The solution for the algebras turns out to be quite different according to the rank (defined in Section 4.2.2) of the fermionic extension, but could be reduced essentially to an algebraic problem—despite the fact that the Jacobi identities represent a set of nonlinear first order differential equations in terms of the target space coordinates.

In this argument the Casimir functions are found to play a key role. If the fermionic extension is of full rank that function of the corresponding bosonic PSM simply generalizes to a quantity C, taking values in the (commuting) supernumbers, because a quadratic contribution of χ^α is included (cf. (4.21)). If the extension is not of full rank, apart from the commuting C also anticommuting Casimir functions of the form (4.52) and (4.53) appear.

In certain cases, but not in general, the use of target space diffeomorphisms (cf. Section 4.3) was found to be a useful tool for the construction of the specific algebras and ensuing supergravity models. The study of 'stabilisators', target space transformations which leave an initially given bosonic algebra invariant, also clarified the large arbitrariness (dependence of the solution on arbitrary functions) found for the Poisson superalgebras and the respective supergravity actions.

Because of this we have found it advisable to study explicit specialized algebras and supergravity theories of increasing complexity (Section 4.4 and 4.6). Our examples are chosen in such a way that the extension of known bosonic 2d models of gravity, like the Jackiw-Teitelboim model [92–96], the dilaton black-hole [59–61, 80–84], spherically symmetric gravity, the Katanaev-Volovich model [54, 55], R^2-gravity and others could be covered (cf. Section 4.6). The arbitrariness referred to above has the consequence that in all cases examples of several possible extensions can be given. For a generic supergravity, obtained in this manner, obstructions for the allowed values of the bosonic target space coordinates emerge. Certain extensions are even found to be not viable within real extensions of the bosonic algebra. We identified two sources of these problems: the division by a certain determinant (cf. (4.47)) in the course of the (algebraic) solution of the Jacobi identities and the appearance of a 'prepotential' which may be nontrivially

related to the potential in the original PSM. Our hopes for the existence of an eventual criterion for a reduction of the inherent arbitrariness, following from the requirement that such obstructions should be absent, unfortunately did not materialize: e.g. for the physically interesting case of an extension of spherically reduced Einstein gravity no less than four different obstruction-free supergravities are among the examples discussed here, and there exist infinitely more.

The PSM approach for 2d gravities contains a preferred formulation of gravity as a 'first order' (in derivatives) action (1.2) in the bosonic, as well as in the supergravity case (Section 4.5).

In this formulation the target space coordinates $X^I = (\phi, X^a, \chi^\alpha)$ of the gPSM are seen to coincide with the momenta in a Hamiltonian action. A Hamiltonian analysis is not pursued in our present work. Instead we discuss the possibility to eliminate X^I in part or completely.

The elimination of X^a is possible in the action of a generic supergravity PSM together with a torsion dependent part of the spin connection. We show that in this way the most general superdilaton theory with usual bosonic part (1.3) and minimal content of fermionic fields in its extension (the Majorana spinor χ^α as partner of the dilaton field ϕ, and the 1-form 'gravitino' ψ_α) is produced.

By contrast, the elimination of the dilaton field ϕ and/or the related spinor χ^α can only be achieved in particular cases. Therefore, these fields should be regarded as substantial ingredients when extending a bosonic 2d gravity action of the form (1.1), depending on curvature and torsion.

The supergravity models whose bosonic part is torsion-free ($Z = 0$ in (1.2) with (1.4), or in (1.3)) have been studied before [129–132]. Specializing the potential $v(\phi)$ *and* the extension appropriately, one arrives at the supersymmetric extension of the R^2-model ($v = -\frac{\alpha}{2}\phi^2$) and the model of Howe ($v = -\frac{1}{2}\phi^3$) [46] originally derived in terms of superfields (cf. also [141]). In the latter case the 'auxiliary field' A is found to coincide with the dilaton field ϕ. It must be emphasized, though, that in the PSM approach all these models are obtained by introducing a cancellation mechanism for singularities and ensuing obstructions (for actions and solutions in the real numbers).

When the bosonic model already contains torsion in the PSM form (1.2) or when, equivalently, $Z \neq 0$ in (1.3) an extension of a conformal transformation to a target space diffeomorphism in the gPSM allows an appropriate generalization of the models with $Z = 0$. Our discussion of spherically symmetric gravity (1.5) and of the Katanaev-Volovich model (1.7) show basic differences. Whereas no new obstruction appears for the former 'physical' theory ($\phi > 0$), as already required by (1.5), the latter model develops a problem with real actions, except when the parameters α, β and Λ are chosen in a very specific manner.

We also present a field theoretic model for a gPSM with rank $(2|1)$,

when only one component of the target space spinor χ^α is involved. Its supersymmetry only contains one anticommuting function so that this class of models can be interpreted as $N = (1, 0)$ supergravity.

Finally (Section 4.7) we make the general considerations of [132] more explicit by giving the full (analytic) solution to the class of models summarized in the two preceding paragraphs. It turns out to be sufficient to discuss the case $Z = 0$, because $Z \neq 0$ can be obtained by conformal transformation. Our formulation in terms of Casimir-Darboux coordinates (including the fermionic extension) allows the integration of the infinitesimal supersymmetry transformation to finite ones. Within the range of applicability for the target space coordinates X^I this permits a gauging of the target space spinors to zero. In this sense supergravities (without matter) are 'trivial'. However, as stressed in the introduction, such arguments break down when (supersymmetric) matter is coupled to the model.

Finally, in Chapter 5 the relations between the superfield approach and the gPSM one are discussed. Taking the Howe model as an explicit example the auxiliary field A in Howe's model is identified with the dilaton field ϕ which appears already in the bosonic theory. In this manner also the relation to the first oder formulation [56–58, 71, 79, 62, 63] is clarified.

This leads us to an outlook on possible further applications. There is a multitude of directions for future work suggested by our present results. We only give a few examples.

Clearly starting from any of the models described here, its supertransformations could be used—at least in a trial-and-error manner as in the original $d = 4$ supergravity [31–35]—to extend the corresponding bosonic action [131].

The even simpler introduction of matter in the form of a scalar 'testparticle' in gravity explicitly (or implicitly) is a necessary prerequisite for defining the global manifold geometrically to its geodesics (including null directions). We believe that a (properly defined) spinning testparticle would be the appropriate instrument for 2d supergravity (cf. e.g. [53]). 'Trivial' supergravity should be without influence on its ('super')-geodesics. This should work in the same way as coordinate singularities are not felt in bosonic gravity.

Another line of investigation concerns the reduction of $d \geq 4$ supergravities to a $d = 2$ effective superdilaton theory. In this way it should perhaps be possible to nail down the large arbitrariness of superdilaton models, when—as in our present work—this problem is regarded from a strictly $d = 2$ point of view. It can be verified in different ways that the introduction of Killing spinors within such an approach inevitably leads to complex fermionic (Dirac) components. Thus 2d (dilatonic) supergravities with $N \geq 2$ must be considered. As explained in Section 4.1 also for this purpose the gPSM approach seems to be the method of choice. Of course, the increase in the number of fields, together with the restrictions of the

additional $SO(N)$ symmetry will provide an even more complicated structure. Already for $N = 1$ we had to rely to a large extent on computer-aided techniques.

Preliminary computations show that the 'minimal' supergravity actions, provided by the PSM approach, also seem to be most appropriate for a Hamiltonian analysis leading eventually to a quantum 2d supergravity, extending the analogous result for a purely bosonic case [57, 79, 86, 71, 87]. The role of the obstruction for real supersymmetric extensions, encountered for some of the models within this thesis, will have to be reconsidered carefully in this context.

Appendix A

Forms and Gravity

We use the characters $a, b, c, \ldots$ to denote Lorentz indices and $m, n, l, \ldots$ to denote world indices, both types take the values $(0, 1)$. For the Kronecker symbol we write $\delta_a{}^b \equiv \delta_a^b$ and for later convenience we also define the generalized Kronecker symbol

$$\hat{\epsilon}_{ab}^{cd} := \delta_a^c \delta_b^d - \delta_a^d \delta_b^c = \hat{\epsilon}_{ab} \hat{\epsilon}^{cd}, \tag{A.1}$$

where $\hat{\epsilon}_{01} \equiv \hat{\epsilon}^{01} := 1$ is the alternating ϵ-symbol.

In $d = 2$ our Minkowski metric is

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.2}$$

and for the antisymmetric ϵ-tensor we set $\epsilon_{ab} := \hat{\epsilon}_{ab}$ and consistently $\epsilon^{ab} = -\hat{\epsilon}^{ab}$, so that

$$\epsilon_{ab} = -\epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.3}$$

It obeys

$$\epsilon_{ab}\epsilon^{cd} = \delta_a{}^d \delta_b{}^c - \delta_a{}^c \delta_b{}^d, \qquad \epsilon_a{}^b \epsilon_b{}^c = \delta_a{}^c, \qquad \epsilon^{ab}\epsilon_{ba} = 2, \tag{A.4}$$

and $\epsilon_a{}^b$ is also the generator of Lorentz transformations in $d = 2$. The totally antisymmetric tensor and the Minkowskian metric satisfy the Fierz-type identity

$$\eta_{ab}\epsilon_{cd} + \eta_{da}\epsilon_{bc} + \eta_{cd}\epsilon_{ab} + \eta_{bc}\epsilon_{da} = 0, \tag{A.5}$$

which allows to make rearrangements in third and higher order monomials of Lorentz vectors.

Let $x^m = (x^0, x^1)$ be local coordinates on a two-dimensional manifold. We define the components of a 1-form λ and a 2-form ϵ according to

$$\lambda = dx^m \lambda_m, \qquad \epsilon = \frac{1}{2} dx^m \wedge dx^n \epsilon_{nm}. \tag{A.6}$$

The exterior derivative of a function f is $df = dx^m \partial_m f$, as customary, but for a 1-form λ we choose

$$d\lambda := dx^m \wedge dx^n \partial_n \lambda_m. \tag{A.7}$$

As a consequence d acts from the *right*, i.e. for a q-form ψ and a p-form ϕ the Leibniz rule is

$$d(\psi \wedge \phi) = \psi \wedge d\phi + (-1)^p d\psi \wedge \phi. \tag{A.8}$$

This convention is advantageous for the extension to spinors and superspace (cf. Section 2.1.1 and Appendix B) where we assume the same summation convention of superindices.

Purely bosonic two-dimensional gravity is described in terms of the anholonomic orthonormal basis in tangent or equivalently in cotangent space,

$$\partial_a = e_a{}^m \partial_m, \qquad e^a = dx^m e_m{}^a, \tag{A.9}$$

where $e_m{}^a$ is the zweibein and $e_a{}^m$ its inverse. We use $\partial_a = e_a{}^m \partial_m$ to denote the moving frame, because e_a is used in the PSM context for the 1-forms $e_a = e^b \eta_{ba}$. The zweibein is an isomorphism which transforms world indices into tangent indices and vice versa, therefore $e := \det(e_m{}^a) = \det(e_a{}^m)^{-1} \neq 0$. In terms of the metric $g_{mn} = e_n{}^b e_m{}^a \eta_{ab}$ and its determinant $g = \det(g_{mn})$ we have $e = \sqrt{-g}$. The relationship $e_a{}^m e_m{}^b = \delta_a{}^b$ can be expressed in $d = 2$ by the closed formulae

$$e_m{}^a = \frac{1}{\det(e_a{}^m)}\, \hat\epsilon^{ab}_{mn} e_b{}^n, \qquad e_a{}^m = \frac{1}{\det(e_m{}^a)}\, \hat\epsilon^{mn}_{ab} e_n{}^b, \tag{A.10}$$

where $\hat\epsilon^{ab}_{mn}$ and $\hat\epsilon^{mn}_{ab}$ are the generalized Kronecker symbols derived from (A.1) by simply rewriting that formula with appropriate indices.

The transition to world indices of the ϵ-tensor (A.3) yields

$$\epsilon_{mn} = e\, \hat\epsilon_{mn}, \qquad \epsilon^{mn} = -\frac{1}{e}\, \hat\epsilon^{mn}. \tag{A.11}$$

The induced volume form

$$\epsilon = \frac{1}{2} e^a \wedge e^b \epsilon_{ba} = e^1 \wedge e^0 = e\, dx^1 \wedge dx^0 \tag{A.12}$$

enables us to derive the useful relation $dx^m \wedge dx^n = \epsilon\, \epsilon^{mn}$, and to define the Hodge dual according to

$$\star 1 = \epsilon, \qquad \star dx^m = dx^n \epsilon_n{}^m, \qquad \star\epsilon = 1. \tag{A.13}$$

It is a linear map, i.e. $\star(\phi f) = (\star\phi) f$ for functions f and forms ϕ, and bijective $\star\star = \mathrm{id}$.

The anholonomicity coefficients $c_{ab}{}^c$ from

$$[\partial_a, \partial_b] = c_{ab}{}^c \partial_c, \qquad de^a = -\frac{1}{2} dx^m \wedge dx^n c_{nm}{}^a \tag{A.14}$$

can be expressed by the zweibein as well as by its inverse according to

$$c_{ab}{}^c = (\partial_a e_b{}^n - \partial_b e_a{}^n) e_n{}^c, \qquad c_{mn}{}^a = -(\partial_m e_n{}^a - \partial_n e_m{}^a). \tag{A.15}$$

In two dimensions the anholonomicity coefficients are in one-to-one correspondence with their own trace

$$c_b = c_{ab}{}^a, \qquad c_{ab}{}^c = \delta_a{}^c c_b - \delta_b{}^c c_a. \tag{A.16}$$

With

$$\tilde{\omega}^b := \epsilon^{nm}(\partial_m e_n{}^b) \tag{A.17}$$

the useful relations

$$\tilde{\omega}^c = \epsilon^{cb} c_b = -\frac{1}{2}\epsilon^{ba} c_{ab}{}^c, \qquad c_{ab}{}^c = -\epsilon_{ab}\tilde{\omega}^c \tag{A.18}$$

and

$$\epsilon^{nm}\partial_m v_n = \epsilon^{ba}(\partial_a v_b - \tilde{\omega}_a \epsilon_b{}^c v_c) = \epsilon^{ba}\partial_a v_b + \tilde{\omega}^c v_c \tag{A.19}$$

are obtained.

Apart from the zweibein a two-dimensional spacetime is characterized by the Lorentz connection $\omega_{ma}{}^b = \omega_m \epsilon_a{}^b$ providing the connection 1-form $\omega = dx^m \omega_m$ and the exterior covariant derivative of vector and covector valued differential forms

$$D\phi^a := d\phi^a + \phi^b \wedge \omega \epsilon_b{}^a, \qquad D\phi_a := d\phi_a - \epsilon_a{}^b \phi_b \wedge \omega. \tag{A.20}$$

The exterior covariant derivative D_m acts exclusively on Lorentz indices, whereas the full covariant derivative ∇_m includes also world indices. The latter is expressed with the help of the linear connection $\Gamma_{mn}{}^l$, e.g. for vectors and covectors the component expressions

$$D_m v^b = \partial_m v^b + v^c \omega_{mc}{}^b, \qquad \nabla_m v^n = \partial_m v^n + v^l \Gamma_{ml}{}^n, \tag{A.21}$$

$$D_m v_b = \partial_m v_b - \omega_{mb}{}^c v_c, \qquad \nabla_m v_n = \partial_m v_n - \Gamma_{mn}{}^l v_l. \tag{A.22}$$

are significant. Consistency for interchanging both types of indices leads to the assertion

$$(D_m v^b) e_b{}^n = \nabla_m v^n \quad \Leftrightarrow \quad \nabla_m e_b{}^n = 0. \tag{A.23}$$

From the last equation we obtain

$$\Gamma_{mn}{}^{l} = (D_m e_n{}^{b}) e_b{}^{l} = (\partial_m e_n{}^{b} + e_n{}^{c} \omega_{mc}{}^{b}) e_b{}^{l}. \tag{A.24}$$

The torsion 2-form and its Hodge dual (cf. (A.13)) are defined by

$$t^a := De^a, \qquad \tau^a := \star t^a. \tag{A.25}$$

For the components of $t^a = \frac{1}{2} dx^m \wedge dx^n t_{nm}{}^{a}$ and for τ^a the expressions

$$t_{ab}{}^{c} = -c_{ab}{}^{c} + \omega_a \epsilon_b{}^{c} - \omega_b \epsilon_a{}^{c}, \qquad \tau^a = \frac{1}{2} \epsilon^{mn} t_{nm}{}^{a} \tag{A.26}$$

are obtained, and due to condition (A.24)

$$t_{mn}{}^{l} = \Gamma_{mn}{}^{l} - \Gamma_{nm}{}^{l}. \tag{A.27}$$

The vector τ^a is frequently encountered in Chapter 4. It immediately provides a simple formula relating torsion and Lorentz connection:

$$\omega_a = \tilde{\omega}_a - \tau_a \tag{A.28}$$

Here $\tilde{\omega}_a$ is the quantity already defined by (A.17) which turns out to be the torsion free connection of Einstein gravity.

The curvature tensor is quite simple in two dimensions:

$$r_{mna}{}^{b} = f_{mn} \epsilon_a{}^{b}, \qquad f_{mn} = \partial_m \omega_n - \partial_n \omega_m \tag{A.29}$$

Expressed in terms of ω_a one obtains

$$f_{ab} = \partial_a \omega_b - \partial_b \omega_a - c_{ab}{}^{c} \omega_c = \nabla_a \omega_b - \nabla_b \omega_a + t_{ab}{}^{c} \omega_c. \tag{A.30}$$

Ricci tensor and scalar read

$$r_{ab} = r_{cab}{}^{c} = \epsilon_a{}^{c} f_{cb}, \qquad r = r_a{}^{a} = \epsilon^{ba} f_{ab}, \tag{A.31}$$

leading to the relation

$$\frac{r}{2} = \star d\omega = \epsilon^{nm} \partial_m \omega_n, \tag{A.32}$$

which is valid exclusively in the two-dimensional case.

Appendix B

Conventions of Spinor-Space and Spinors

Of course, the properties of Clifford algebras and spinors in any number of dimensions (including $d = 1+1$) are well-known, but in view of the tedious calculations required in our present work we include this appendix in order to prevent any misunderstandings of our results and facilitate the task of the intrepid reader who wants to redo derivations.

The γ matrices which are the elements of the Clifford algebra defined by the relation

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbb{1}, \qquad \eta_{ab} = \eta^{ab} = \mathrm{diag}(+-), \tag{B.1}$$

are represented by two-dimensional matrices

$$\gamma^0{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{B.2}$$

As indicated, the lower index is assumed to be the first one. The spinor indices are often suppressed assuming the summation from 'ten to four'. The generator of a Lorentz boost (hyperbolic rotation) has the form

$$\sigma^{ab} = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a) = \epsilon^{ab}\gamma^3, \tag{B.3}$$

where

$$\gamma^3 = \gamma^1\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\gamma^3)^2 = \mathbb{1}. \tag{B.4}$$

In two dimensions the γ-matrices satisfy the relation

$$\gamma^a\gamma^b = \eta^{ab}\mathbb{1} + \epsilon^{ab}\gamma^3, \tag{B.5}$$

which is equivalent to the definition (B.1). The following formulae are frequently used in our calculations:

$$\begin{aligned} \gamma^a\gamma_a &= 2, & \gamma^a\gamma^b\gamma_a &= 0, \\ \gamma^a\gamma^3 + \gamma^3\gamma^a &= 0, & \gamma^a\gamma^3 &= \gamma^b\epsilon_b{}^a, \\ \mathrm{tr}(\gamma^a\gamma^b) &= 2\eta^{ab}. \end{aligned} \tag{B.6}$$

As usual the trace of the product of an odd number of γ-matrices vanishes.

In two dimensions the γ-matrices satisfy the Fierz identity

$$\begin{aligned} 2\gamma^a{}_\alpha{}^\gamma\gamma^b{}_\beta{}^\delta &= \gamma^a{}_\alpha{}^\delta\gamma^b{}_\beta{}^\gamma + \gamma^b{}_\alpha{}^\delta\gamma^a{}_\beta{}^\gamma + \\ &+ \eta^{ab}(\delta_\alpha{}^\delta\delta_\beta{}^\gamma - \gamma^3{}_\alpha{}^\delta\gamma^3{}_\beta{}^\gamma - \gamma^c{}_\alpha{}^\delta\gamma_{c\beta}{}^\gamma) + \epsilon^{ab}(\gamma^3{}_\alpha{}^\delta\delta_\beta{}^\gamma - \delta_\alpha{}^\delta\gamma^3{}_\beta{}^\gamma), \end{aligned} \tag{B.7}$$

which can be checked by direct calculation. Different contractions of it with γ-matrices then yield different but equivalent versions

$$2\delta_\alpha{}^\gamma\delta_\beta{}^\delta = \delta_\alpha{}^\delta\delta_\beta{}^\gamma + \gamma^3{}_\alpha{}^\delta\gamma^3{}_\beta{}^\gamma + \gamma^a{}_\alpha{}^\delta\gamma_{a\beta}{}^\gamma, \tag{B.8}$$

$$2\gamma^3{}_\alpha{}^\gamma\gamma^3{}_\beta{}^\delta = \delta_\alpha{}^\delta\delta_\beta{}^\gamma + \gamma^3{}_\alpha{}^\delta\gamma^3{}_\beta{}^\gamma - \gamma^a{}_\alpha{}^\delta\gamma_{a\beta}{}^\gamma, \tag{B.9}$$

$$\gamma^a{}_\alpha{}^\gamma\gamma_{a\beta}{}^\delta = \delta_\alpha{}^\delta\delta_\beta{}^\gamma - \gamma^3{}_\alpha{}^\delta\gamma^3{}_\beta{}^\gamma, \tag{B.10}$$

which allow to manipulate third and higher order monomials in spinors. Notice that equation (B.5) is also the consequence of (B.7). An other manifestation of the Fierz identity is the completeness relation

$$\Gamma_\alpha{}^\beta = \frac{1}{2}\Gamma_\gamma{}^\gamma\,\delta_\alpha{}^\beta + \frac{1}{2}(\Gamma\gamma_a)_\gamma{}^\gamma\,\gamma^a{}_\alpha{}^\beta + \frac{1}{2}(\Gamma\gamma^3)_\gamma{}^\gamma\,\gamma^3{}_\alpha{}^\beta. \tag{B.11}$$

A Dirac spinor in two dimensions, forming an irreducible representation for the full Lorentz group including space and time reflections, has two complex components. We write it—in contrast to the usual convention in field theory, but in agreement with conventional superspace notations—as a row

$$\chi^\alpha = (\chi^+, \chi^-). \tag{B.12}$$

In our notation the first and second components of a Dirac spinor correspond to right and left chiral Weyl spinors $\chi^{(\pm)}$, respectively,

$$\chi^{(\pm)} = \chi P_\pm, \qquad \chi^{(\pm)}\gamma^3 = \pm\chi^{(\pm)}, \tag{B.13}$$

where the chiral projectors are given by

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma^3) \tag{B.14}$$

so that

$$\chi^{(+)} = (\chi^+, 0), \qquad \chi^{(-)} = (0, \chi^-). \tag{B.15}$$

Here matrices act on spinors form the right according to the usual multiplication law. All spinors are always assumed to be anticommuting variables. The notation with upper indices is a consequence of our convention to contract indices, together with the usual multiplication rule for matrices. Under the Lorentz boost by the parameter ω spinors transform as

$$\chi'^{\alpha} = \chi^{\beta} S_{\beta}{}^{\alpha}, \tag{B.16}$$

where

$$S_{\beta}{}^{\alpha} = \delta_{\beta}{}^{\alpha} \cosh\frac{\omega}{2} - \gamma^{3}{}_{\beta}{}^{\alpha} \sinh\frac{\omega}{2} = \begin{pmatrix} e^{-\omega/2} & 0 \\ 0 & e^{+\omega/2} \end{pmatrix}, \tag{B.17}$$

when the Lorentz boost of a vector is given by the matrix

$$S_{b}{}^{a} = \delta_{b}{}^{a} \cosh\omega + \epsilon_{b}{}^{a} \sinh\omega = \begin{pmatrix} \cosh\omega & \sinh\omega \\ \sinh\omega & \cosh\omega \end{pmatrix}. \tag{B.18}$$

By (B.17), (B.18) the γ-matrices are invariant under simultaneous transformation of Latin and Greek indices. This requirement fixes the relative factors in the bosonic and fermionic sectors of the Lorentz generator.

For a spinor

$$\chi_{\alpha} = \begin{pmatrix} \chi_{+} \\ \chi_{-} \end{pmatrix} \tag{B.19}$$

the Dirac conjugation $\bar{\chi}^{\alpha} = \chi^{\dagger\dot{\alpha}} A_{\dot{\alpha}}{}^{\alpha}$ depends on the matrix A, which obeys

$$A\gamma^{a}A^{-1} = (\gamma^{a})^{\dagger}, \qquad A^{\dagger} = A. \tag{B.20}$$

We make the usual choice

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^{0}. \tag{B.21}$$

The charge conjugation of a spinor using complex conjugation is

$$\chi^{c} = B\chi^{*} \tag{B.22}$$

$$B^{-1}\gamma^{a}B = -(\gamma^{a})^{*}, \qquad BB^{*} = \mathbb{1}. \tag{B.23}$$

For our choice of γ^{a} (B.2)

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.24}$$

Alternatively, one can define the charge conjugated spinor with the help of the Dirac conjugation matrix (B.21),

$$\chi^{c} = (\bar{\chi}C)^{T} = C^{T}A^{T}\chi^{*}, \qquad \chi^{c}_{\alpha} = \bar{\chi}^{\beta}C_{\beta\alpha}, \tag{B.25}$$

$$C^{-1}\gamma^a C = -(\gamma^a)^T, \qquad C^T = -C, \tag{B.26}$$

$$C = (C_{\beta\alpha}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{B.27}$$

By means of

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{B.28}$$

indices of Majorana spinors $\chi^c = \chi$, in components $\chi_+ = -(\chi_+)^*, \chi_- = (\chi_-)^*$, can be raised and lowered as $\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta$ and $\chi_\alpha = \chi^\beta\epsilon_{\beta\alpha}$. In components we get

$$\chi^+ = \chi_-, \qquad \chi^- = -\chi_+. \tag{B.29}$$

This yields $\psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \psi^-\chi^+ - \psi^+\chi^-$ for two anticommuting Majorana spinors ψ and χ. For bilinear forms we use the shorthand

$$(\psi\chi) = \psi^\alpha\chi_\alpha, \qquad (\psi\gamma^a\chi) = \psi^\alpha\gamma^a{}_\alpha{}^\beta\chi_\beta, \qquad (\psi\gamma^3\chi) = \psi^\alpha\gamma^3{}_\alpha{}^\beta\chi_\beta. \tag{B.30}$$

A useful property is

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = \delta_\alpha{}^\gamma\delta_\beta{}^\delta - \delta_\alpha{}^\delta\delta_\beta{}^\gamma. \tag{B.31}$$

The Fierz identity (B.8) yields

$$\chi_\alpha\psi^\beta = -\frac{1}{2}(\psi\chi)\delta_\alpha{}^\beta - \frac{1}{2}(\psi\gamma_a\chi)\gamma^a{}_\alpha{}^\beta - \frac{1}{2}(\psi\gamma^3\chi)\gamma^3{}_\alpha{}^\beta. \tag{B.32}$$

Among the spinor matrices $\gamma^{a\alpha\beta}$ and $\gamma^{3\alpha\beta}$ are symmetric in $\alpha \leftrightarrow \beta$, whereas $\epsilon^{\alpha\beta}$ is antisymmetric.

B.1 Spin-Components of Lorentz Tensors

The light cone components of a Lorentz vector $v^a = (v^0, v^1)$ are defined according to

$$v^\oplus = \frac{1}{\sqrt{2}}\left(v^0 + v^1\right), \qquad v^\ominus = \frac{1}{\sqrt{2}}\left(v^0 - v^1\right). \tag{B.33}$$

We use the somewhat unusual symbols $\oplus$ and $\ominus$ to denote light cone indices, in order to avoid confusion with spinor indices. The Lorentz metric is off-diagonal in this basis, $\eta_{\oplus\ominus} = \eta_{\ominus\oplus} = 1$, leading to the usual property for raising and lowering indices $v^\oplus = v_\ominus$ and $v^\ominus = v_\oplus$. The nonzero components of the epsilon tensor are $\epsilon_{\oplus\ominus} = -1$ and $\epsilon_{\ominus\oplus} = 1$. When raising the last

index the generator of Lorentz transformation $\epsilon_a{}^b$ is found to be diagonal, $\epsilon_\oplus{}^\oplus = -1$ and $\epsilon_\ominus{}^\ominus = 1$. For quadratic forms we obtain

$$v^a w^b \eta_{ba} = v^\oplus w^\ominus + v^\ominus w^\oplus, \qquad v^a w^b \epsilon_{ba} = v^\oplus w^\ominus - v^\ominus w^\oplus. \tag{B.34}$$

In theories with spinors, and thus in the presence of γ-matrices, it is convenient to associate to every Lorentz vector v^a a spin-tensor according to $v^{\alpha\beta} = a v^c \gamma_c{}^{\alpha\beta}$ with the convenient value of the constant a to be determined below. In $d = 2$ this spin-tensor is symmetric ($v^{\alpha\beta} = v^{\beta\alpha}$) and γ^3-traceless ($v^{\alpha\beta}\gamma^3{}_{\beta\alpha} = 0$). In order to make these properties manifest we define the projectors

$$T^{\langle\alpha\beta\rangle} := -\frac{1}{2} T^{\gamma\delta}\gamma^a{}_{\delta\gamma}\gamma_a{}^{\alpha\beta} = T^{(\alpha\beta)} + \frac{1}{2} T^{\gamma\delta}\gamma^3{}_{\delta\gamma}\gamma^{3\alpha\beta}, \tag{B.35}$$

thus $v^{\alpha\beta} = v^{\langle\alpha\beta\rangle}$. We use angle brackets to express explicitly the fact that a spin-pair originates from a Lorentz vector and write $v^{\langle\alpha\beta\rangle}$ from now on. In the chosen representation of the γ-matrices mixed components vanish, $v^{\langle+-\rangle} = v^{\langle-+\rangle} = 0$. Furthermore, as a consequence of the spinor metric (B.28) the spin pairs $\langle++\rangle$ and $\langle--\rangle$ behave exactly as light cone indices; these indices are lowered according to

$$v^{\langle++\rangle} = v_{\langle--\rangle}, \qquad v^{\langle--\rangle} = v_{\langle++\rangle}. \tag{B.36}$$

In order to be consistent the Lorentz metric in spinor components has to be of light cone form too: By definition we have $\eta_{\langle\alpha\beta\rangle\langle\delta\gamma\rangle} = a^2 \gamma^a{}_{\alpha\beta}\gamma^b{}_{\delta\gamma}\eta_{ab}$, which is indeed off diagonal and symmetric when interchanging the spin-pairs. Demanding $\eta_{\langle++\rangle\langle--\rangle} = 1$ fixes the parameter a up to a sign, yielding $a = \pm\frac{i}{\sqrt{2}}$. We choose the positive sign yielding for the injection of vectors into spin-tensors, and for the extraction of vectors from spin-tensors

$$v^{\langle\alpha\beta\rangle} := \frac{i}{\sqrt{2}} v^a \gamma_a{}^{\alpha\beta}, \qquad T^a := \frac{i}{\sqrt{2}} T^{\alpha\beta}\gamma^a{}_{\beta\alpha}, \tag{B.37}$$

respectively. This definition leads to

$$\eta_{\langle\alpha\beta\rangle\langle\delta\gamma\rangle} = -\frac{1}{2}\gamma^a{}_{\alpha\beta}\gamma_{a\delta\gamma}, \tag{B.38}$$

$$\gamma_{\langle\alpha\beta\rangle\delta\gamma} = \frac{i}{\sqrt{2}}\gamma^a{}_{\alpha\beta}\gamma_{a\delta\gamma} = -i\sqrt{2}\eta_{\langle\alpha\beta\rangle\langle\delta\gamma\rangle} \tag{B.39}$$

for the conversion of Lorentz indices in η_{ab} and $\gamma_{a\delta\gamma}$ to spin-pairs, yielding for the latter also a spin-tensor which is symmetric when interchanging the spin pairs, $\gamma_{\langle\alpha\beta\rangle\langle\delta\gamma\rangle} = \gamma_{\langle\delta\gamma\rangle\langle\alpha\beta\rangle}$.

The spin pair components are not identical to the light cone ones as defined above in (B.33). The former are purely imaginary and related to the latter according to

$$v^{\langle++\rangle} = i v^\oplus, \qquad v^{\langle--\rangle} = -i v^\ominus. \tag{B.40}$$

The components of the epsilon tensor are $\epsilon_{\langle ++\rangle\langle --\rangle} = -1$ and $\epsilon_{\langle --\rangle\langle ++\rangle} = 1$ and the quadratic forms read

$$v^a w_a = v^{\langle\alpha\beta\rangle} w_{\langle\beta\alpha\rangle} = v^{\langle ++\rangle} w^{\langle --\rangle} + v^{\langle --\rangle} w^{\langle ++\rangle}, \tag{B.41}$$

$$v^a w^b \epsilon_{ba} = v^{\langle\alpha\beta\rangle} w^{\langle\gamma\delta\rangle} \epsilon_{\langle\delta\gamma\rangle\langle\beta\alpha\rangle} = v^{\langle ++\rangle} w^{\langle --\rangle} - v^{\langle --\rangle} w^{\langle ++\rangle}. \tag{B.42}$$

Note that we can exchange contracted Lorentz indices with spin pair indices in angle brackets—a consequence of our choice for parameter a.

The metric $\eta_{\langle\alpha\beta\rangle\langle\gamma\delta\rangle}$ is only applicable to lower spin-pair indices which correspond to Lorentz vectors. It can be extended to a full metric $\eta_{\alpha\beta}{}^{\delta\gamma}$, demanding $T^{\beta\alpha}\eta_{\alpha\beta}{}^{\delta\gamma} = T^{\delta\gamma}$ for any spin-tensor $T^{\beta\alpha}$. This implies the γ-matrix expansion

$$\eta_{\alpha\beta}{}^{\delta\gamma} = -\frac{1}{2}\gamma^a{}_{\alpha\beta}\gamma_a{}^{\delta\gamma} - \frac{1}{2}\gamma^3{}_{\alpha\beta}\gamma^{3\delta\gamma} - \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{\delta\gamma}. \tag{B.43}$$

Using Fierz identity (B.8) we find the simple formula

$$\eta_{\alpha\beta}{}^{\delta\gamma} = \delta_\alpha{}^\gamma \delta_\beta{}^\delta. \tag{B.44}$$

For easier writing we omit in the main text of this work the brackets when adequate ($v^{\langle ++\rangle} \to v^{++}$, $v^{\langle --\rangle} \to v^{--}$, etc.).

B.2 Decompositions of Spin-Tensors

Any symmetric spinor $W_{\alpha\beta} = W_{\beta\alpha}$ can be decomposed in a vector and a pseudoscalar

$$W_{\alpha\beta} = W^a \gamma_{a\alpha\beta} + W^3 \gamma^3{}_{\alpha\beta} \tag{B.45}$$

given by

$$W^a = \frac{1}{2}(W\gamma^a)_\gamma{}^\gamma, \qquad W^3 = \frac{1}{2}(W\gamma^3)_\gamma{}^\gamma. \tag{B.46}$$

The symmetric tensor product of two spinors is therefore

$$\chi_\alpha\psi_\beta + \chi_\beta\psi_\alpha = (\chi\gamma_a\psi)\,\gamma^a{}_{\alpha\beta} + (\chi\gamma^3\psi)\,\gamma^3{}_{\alpha\beta}. \tag{B.47}$$

The field $\psi^{a\alpha}$ has one vector and one spinor index. We assume that for each a it is a Majorana spinor. Therefore it has two real and two purely imaginary components forming a reducible representation of the Lorentz group. In many applications it becomes extremely useful to work with its Lorentz covariant decomposition

$$\psi^a = \psi\gamma^a + \lambda^a, \tag{B.48}$$

where

$$\psi = \frac{1}{2}(\psi^a \gamma_a), \qquad \lambda^a = \frac{1}{2}(\psi^b \gamma^a \gamma_b) = \frac{1}{2}(\gamma_b \gamma^a \psi^b). \tag{B.49}$$

The spinor ψ^α and the spin-vector ${\lambda_a}^\alpha$ form irreducible representations of the Lorentz group and each of them has two independent components. The spin-vector λ_a satisfies the Rarita-Schwinger condition

$$\lambda_a \gamma^a = 0 \tag{B.50}$$

valid for such a field. In two dimensions equation (B.50) may be written in equivalent forms

$$\lambda_a \gamma_b = \lambda_b \gamma_a \quad \text{or} \quad \epsilon^{ab} \lambda_a \gamma_b = 0. \tag{B.51}$$

If one chooses the $\lambda^{0\alpha}$ components as independent ones then the components of $\lambda^{1\alpha}$ can be found from (B.50) to be

$$\lambda^{0\alpha} = (\lambda^{0+}, \lambda^{0-}), \qquad \lambda^{1\alpha} = (\lambda^{0+}, -\lambda^{0-}).$$

It is important to note that as a consequence any cubic or higher monomial of the (anticommuting) ψ or λ_a vanishes identically. Furthermore, the field λ_a satisfies the useful relation

$${\epsilon_a}^b \lambda_b = \lambda_a \gamma^3 \tag{B.52}$$

which together with (B.6) yields

$${\epsilon_a}^b \psi_b = -\psi \gamma_a \gamma^3 + \lambda_a \gamma^3. \tag{B.53}$$

For the sake of brevity we often introduce the obvious notations

$$\psi^2 = \psi^\alpha \psi_\alpha, \qquad \lambda^2 = \lambda^{a\alpha} \lambda_{a\alpha}. \tag{B.54}$$

Other convenient identities used for λ_a in our present work are

$$(\lambda_a \lambda_b) = \frac{1}{2} \eta_{ab} \lambda^2, \tag{B.55}$$

$$(\lambda_a \gamma^3 \lambda_b) = \frac{1}{2} \epsilon_{ab} \lambda^2, \tag{B.56}$$

$$(\lambda_a \gamma_c \lambda_b) = 0. \tag{B.57}$$

The first of these identities can be proved by inserting the unit matrix $\gamma_a \gamma^a/2$ inside the product and interchanging the indices due to equation (B.51). The second and third equation is antisymmetric in indices a, b, and therefore to be calculated easily because they are proportional to ϵ_{ab}.

Quadratic combinations of the vector-spinor field can be decomposed in terms of irreducible components:

$$(\psi_a \psi_b) = \eta_{ab} \left(-\psi^2 + \frac{1}{2}\lambda^2 \right) + 2(\psi \gamma_a \lambda_b) \tag{B.58}$$

$$(\psi_a \gamma^3 \psi_b) = \epsilon_{ab} \left(\psi^2 + \frac{1}{2}\lambda^2 \right) \tag{B.59}$$

$$(\psi_a \gamma_c \psi_b) = 2\epsilon_{ab} (\psi \gamma^3 \lambda_c) \tag{B.60}$$

$$(\psi_a \gamma_c \gamma^3 \psi_b) = 2\epsilon_{ab} (\psi \lambda_c) \tag{B.61}$$

Any totally symmetric spinor $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ can be written as

$$W_{\alpha\beta\gamma} = \gamma^a{}_{\alpha\beta} W_{a\gamma} + \gamma^a{}_{\beta\gamma} W_{a\alpha} + \gamma^a{}_{\gamma\alpha} W_{a\beta}, \tag{B.62}$$

where the decomposition of the vector-spinor $W_{a\gamma} = W^+{}_{a\gamma} + W^-{}_{a\gamma}$ is given by

$$W^+{}_{a\delta} = -\frac{1}{4} (\gamma_a \gamma^b)_\delta{}^\gamma \gamma_b{}^{\beta\alpha} W_{\alpha\beta\gamma}, \tag{B.63}$$

$$W^-{}_{a\delta} = -\frac{1}{12} (\gamma^b \gamma_a)_\delta{}^\gamma \gamma_b{}^{\beta\alpha} W_{\alpha\beta\gamma}. \tag{B.64}$$

The part $W^+{}_{a\gamma} = -\gamma_{a\gamma}{}^\delta W^+{}_\delta$ can be calculated, with help of the identity

$$\gamma^b{}_\delta{}^{(\gamma} \gamma_b{}^{\beta\alpha)} = -\gamma^3{}_\delta{}^{(\gamma} \gamma^{3\beta\alpha)}, \tag{B.65}$$

by the formula

$$W^+{}_\delta = \frac{1}{4} \gamma^b{}_\delta{}^\gamma \gamma_b{}^{\beta\alpha} W_{\alpha\beta\gamma} = -\frac{1}{4} \gamma^3{}_\delta{}^\gamma \gamma^{3\beta\alpha} W_{\alpha\beta\gamma}. \tag{B.66}$$

If a totally symmetric spinor $V_{\alpha\beta\gamma} = V_{(\alpha\beta\gamma)}$ is given in the form

$$V_{\alpha\beta\gamma} = \gamma^3{}_{\alpha\beta} V_\gamma + \gamma^3{}_{\beta\gamma} V_\alpha + \gamma^3{}_{\gamma\alpha} V_\beta, \tag{B.67}$$

the representation of $V_{\alpha\beta\gamma}$ in the form (B.62) is given by

$$W^+{}_\delta = (\gamma^3 V)_\delta, \qquad W^-{}_{a\delta} = 0, \tag{B.68}$$

therefore

$$V_{\alpha\beta\gamma} = -\gamma^a{}_{\alpha\beta} (\gamma_a \gamma^3 V)_\gamma - \gamma^a{}_{\beta\gamma} (\gamma_a \gamma^3 V)_\alpha - \gamma^a{}_{\gamma\alpha} (\gamma_a \gamma^3 V)_\beta. \tag{B.69}$$

B.3 Properties of the Γ-Matrices

There are several useful brackets the of γ-matrices with Γ^a and Γ^3 which have been used in Section 2.5.4:

$$\{\Gamma^3, \gamma^3\} = 2 \cdot \mathbb{1} \qquad [\Gamma^3, \gamma^3] = -2ik^a \epsilon_a{}^b \gamma_b \tag{B.70}$$

$$\{\Gamma^3, \gamma_a\} = 2ik_a \mathbb{1} \qquad [\Gamma^3, \gamma_a] = 2\epsilon_a{}^b \Gamma_b \tag{B.71}$$

$$\{\Gamma^a, \gamma^3\} = -2ik^a \mathbb{1} \qquad [\Gamma^a, \gamma^3] = [\gamma^a, \gamma^3] = 2\gamma^a\gamma^3 \tag{B.72}$$

$$\{\Gamma^a, \gamma^b\} = 2\eta^{ab} \mathbb{1} \qquad [\Gamma^a, \gamma^b] = 2\epsilon^{ab}\gamma^3 + 2ik^a(\gamma^b\gamma^3) \tag{B.73}$$

The algebra of the field dependent Γ-matrices reads

$$\{\Gamma^3, \Gamma^3\} = 2(1-k^2)\mathbb{1}, \qquad [\Gamma^3, \Gamma^3] = 0, \tag{B.74}$$

$$\{\Gamma^3, \Gamma_a\} = 0, \qquad [\Gamma^3, \Gamma_a] = 2(\delta_a{}^b - k_a k^b)\epsilon_b{}^c \Gamma_c, \tag{B.75}$$

$$\{\Gamma^a, \Gamma^b\} = 2\left(\eta^{ab} - k^a k^b\right)\mathbb{1}, \qquad [\Gamma^a, \Gamma^b] = 2\epsilon^{ab}\Gamma^3, \tag{B.76}$$

where the abbreviation $k^2 = k^a k_a$ was introduced. The anticommutator in (B.76) suggests the interpretation of $\eta^{ab} - k^a k^b$ as a metric in tangent space (cf. for a similar feature [30]). From the algebra

$$\Gamma^3\Gamma^3 = (1-k^2)\mathbb{1}, \tag{B.77}$$

$$\Gamma^3\Gamma_a = (\delta_a{}^b - k_a k^b)\epsilon_b{}^c \Gamma_c, \tag{B.78}$$

$$\Gamma^a\Gamma^b = (\eta^{ab} - k^a k^b)\mathbb{1} + \epsilon^{ab}\Gamma^3 \tag{B.79}$$

is derived. Note that because of $(\delta_a{}^b - k_a k^b)\epsilon_b{}^c = \epsilon_a{}^b\left((1-k^2)\delta_b{}^c + k_b k^c\right)$ we can also write

$$\Gamma^3\Gamma_a = (1-k^2)\epsilon_a{}^b\Gamma_b + k_b k^c \Gamma_c = (1-k^2)\epsilon_a{}^b\gamma_b - i\epsilon_a{}^b k_b \Gamma^3. \tag{B.80}$$

For three products of Γ^a the symmetry property $\Gamma^a\Gamma^b\Gamma^c = \Gamma^c\Gamma^b\Gamma^a$ is valid.

Contractions of the Rarita-Schwinger field with Γ^a similar to (B.49) yield

$$\frac{1}{2}(\psi_b\Gamma_c\Gamma^b)^\alpha = \lambda_b{}^\alpha - \frac{i}{2}k^b(\psi_b\Gamma_c\gamma^3)^\alpha + ik^c(\psi\gamma^3)^\alpha, \tag{B.81}$$

$$\frac{1}{2}(\psi_b\Gamma^b)^\alpha = \psi^\alpha - \frac{i}{2}k^b(\psi_b\gamma^3)^\alpha. \tag{B.82}$$

Some further formulae used in the calculations for terms quadratic in the Rarita-Schwinger field are

$$(\delta_a{}^d - k_a k^d)(\psi_d\Gamma^c\Gamma_b\psi_c) = (\delta_b{}^d - k_b k^d)(\psi_d\Gamma^c\Gamma_a\psi_c), \tag{B.83}$$

$$(\psi_c\Gamma^c\Gamma^3\Gamma^b\Gamma^a\psi_b) = (\psi_c\Gamma^a\Gamma^3\Gamma^b\Gamma^c\psi_b) = (1-k^2)\epsilon^{cb}(\psi_b\Gamma^a\psi_c), \tag{B.84}$$

$$\epsilon^{cb}(\psi_b\Gamma^a\psi_c) = (\psi_b\gamma^a\gamma^3\Gamma^c\Gamma^b\psi_c) = 4(\psi\gamma^3\lambda^a) - ik^a(2\psi^2 + \lambda^2), \tag{B.85}$$

and in the emergence of two different spin-vectors

$$(\psi_b\Gamma^a\gamma_m\varphi^b) = (\varphi^b\gamma_m\Gamma^a\psi_b), \qquad (\psi_b\Gamma^a\gamma^3\varphi^b) = (\varphi^b\gamma^3\Gamma^a\psi_b). \tag{B.86}$$

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